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A Method for Integrating a Set of Ordinary Differential Equations Subject to a Type of Numerical Indeterminacy

1. Introduction. This paper proposes the numerical integration of a system of first order ordinary differential equations by means of a convergent sequence of equation systems. The schema suggested may prove useful whenever: (a) the equations of the set are coupled in an appropriate fashion; (b) ordinary point by point procedures fail due to subtraction which makes the analytic formula for the derivative of one of the variables numerically indeterminate.

The technique is adapted to machine use. On a C.P.C. it has solved a set of four simultaneous non-linear equations which describe a hypothetical one-dimensional free-radical flame [1]; on an IBM 701 it has integrated the seven simultaneous non-linear equations for a hydrogen-bromine flame.

2. Character of Equation Systems for Which the Method may be Useful. Certain physical problems such as free-radical flames are described by sets of simultaneous differential equations in which the calculation of one of the derivatives from its analytic formula involves serious loss of numerical accuracy due to subtraction. The conventional solution of such problems presumes the steady state hypothesis which replaces the differential equation by the algebraic approximation that the derivative vanishes identically. Unfortunately, questions such as the importance of free-radical diffusion in flames must be studied without the *ad hoc* assumption of the steady state. (When applied to free-radical flames, this approximation states that the derivative, dG_B/dt, of the fractional mass rate of flow of the free-radical B vanishes identically. Such an assumption must not be introduced but should be proved valid if we seek to establish the importance of diffusion.) Furthermore, it is generally important to determine the accuracy of such an approximation.

An outline of the qualitative features of one type of equation system which presents this numerical problem will clarify the essential character of the difficulty one meets when he applies traditional point-by-point procedures without assuming the steady state. Consider, therefore, a set of M simultaneous differential equations in an independent variable Z,

$$du_j(Z)/dZ = F_j,\ 1 \leq j \leq M, \tag{1}$$

where in general the F_j may be functions of any or all members of the set of dependent variables $\{u_l\}$ and, or, the independent variable Z. Suppose that the set of functions $\{F_j\}$ has four qualitative characteristics which will be enumerated presently. The first two of these four conditions state that the first function, F_1, is poorly determined numerically. As subsequent discussion explains, the combination of the four conditions describes a coupling which should cause traditonal point by point procedures of integration to fail and which suggests the particular successive approximation schema outlined in Section III.

The conditions are:

$$F_1 = \sum_k a_{1,k}, \qquad F_2 = \sum_k a_{2,k}. \tag{2}$$

(a) F_1 and F_2 are each given as a sum of terms.

(b) F_1 is smaller than one of the terms of equation (2), say, $a_{1,1}$, by a factor of 10^{-p}: $F_1/a_{1,1} = 10^{-p}$, p a positive integer.

(c) Calculation of an s digit value for one of the terms of F_2, say, $a_{2,1}$, requires that u_1 be known to s digits. Moreover, $a_{2,1}$ and F_2 are both of the same numerical magnitude.

(d) An s digit value for the term $a_{1,1}$ numerically determines an s digit value for u_2 (e.g., $a_{1,1}$ might be given analytically as $a_{1,1} = u_2 f$ where f is a function of other variables whose value can be computed accurately to s digits).

Since the conventional methods of computing the increments

$$\Delta_1 u_j(Z_n) = u_j(Z_n) - u_j(Z_{n-1}) \tag{3}$$

in the course of a point by point numerical integration require calculating the first derivatives, F_j, it follows from (a) and (b) that $F_1[\{u_l\}, Z_n]$ and therefore $\Delta_1 u_1(Z_n)$ can be computed to only $(s - p)$ digits whenever the $\{u_l(Z_n)\}$ are known to s digits. (This discussion assumes that an implicit method is being used in the integration. If an explicit method such as the Runge-Kutta were used, the sequence of subscripts on Z would be different, although the principles would be the same.) Furthermore, the error in $\Delta_1 u_1(Z_n)$ appears in $u_1(Z_n)$ and, therefore, by condition (c), in $F_2[\{u_j\}, Z_n]$. Unfortunately this introduces error first into $\Delta_1 u_2(Z_n)$ and thence into $u_2(Z_n)$. The completion of a vicious cycle through which this error reappears in the calculation of $F_1[\{u_l\}, Z_n]$ follows from conditions (b) and (d).

If the desired solution of the equation system actually does maintain conditions (b), (c), and (d) over an appreciable interval in Z, then these conventional methods might be expected to fail completely. The feedback of error might be expected to destroy quickly the approximate equality

$$F_1[\{u_l\}, Z] = 0. \tag{4}$$

In the case of the flame equations for one hypothetical free-radical system, the build-up of error was so rapid that in eight digit calculations the solution was lost in as few as six integration steps. (See Appendix A of reference [1] for an illustration of the failure of conventional techniques. A heuristic interpretation of this failure concludes the appendix.)

3. Proposed Iterative Solution.

A. *General basis of the method.* If the analytic formula for the second derivative of u_1

$$d^2u_1(Z)/dZ^2 = dF_1/dZ \tag{5}$$

did not present the same computational problem, the equation system could presumably be solved by the simple expedient of adopting F_1 as a variable and adding the differential equation for F_1 to the system. However, if the approximation $F_1 \cong 0$ holds for any appreciable range, then the calculation of dF_1/dZ might

be expected to exhibit the same numerical indeterminacy as the calculation of F_1 itself. This is the case in several flame problems.

The very failure of an attempt to circumvent the difficulty by adding the equation for dF_1/dZ to the set suggests that perhaps the approximation $F_1 \cong 0$ is maintained over a considerable range in Z. In this case the four conditions (a)–(d) of Section 2 suggest that the solution to the complete set of equations might be approached by integrating the following sequence of equation systems. The kth approximating set of equations will make use of $(k - 1)$st order approximations for u_1 and du_1/dt. For $k = 1$, the steady state approximation,

(6) $$F_1[\{u_i\}, Z] = 0,$$

is made. According to conditions (b) and (d), equation (6) gives a numerically determinate implicit formula for u_2.

An approximation for $u_1^{(0)}$ can be obtained as follows. Recall that the hypothesis has been made that the analytic expression for dF_1/dZ exhibits the same numerical indeterminacy as F_1. Since, moreover, u_2 is supposed to make a numerically important contribution to $a_{1,1}$, which in turn has been supposed to be one of the most important terms of F_1, it is reasonable to hope that the term

(7) $$(\partial a_{1,1}/\partial u_2)(du_2/dZ) = (\partial a_{1,1}/\partial u_2)F_2$$

will make a numerically important contribution to dF_1/dZ and that a value for dF_1/dZ will determine without significant loss in accuracy a value for F_2. Now, according to condition (c), an s digit value for F_2 is supposed to determine an s digit value for u_1. Thus the approximation

(8) $$dF_1/dZ = 0$$

may provide a numerically determinate implicit equation for F_2 and therefore for $u_1^{(0)}$. These approximations are valid in some free-radical flame systems.

B. *Outline of the steps in the iteration scheme.*

1. *Step for* $k = 1$.

Solve the first set of approximate equations:

(9a) $F_1[\{u_i\}, Z] = 0$ (an implicit algebraic equation for $u_2^{(1)}$)

(9b) $(dF_1/dZ) = 0$ (an implicit algebraic equation for $u_1^{(0)}$)

(9c) $$(du_j(Z)/dZ) = F_j, \qquad 3 \le j \le M.$$

2. *Step for* $(k + 1)$.

Assume that the $u_j^{(k)}$, $2 \le j \le M$ have been computed.

(a) Numerically differentiate $u_2^{(k)}$ to obtain $(du_2/dZ)^{(k)}$.

(b) Since the $u_j^{(k)}$, $2 \le j \le M$ are known, calculate $u_1^{(k)}$ from the equation

(10) $$(du_2/dZ)^{(k)} = F_2[\{u_i^{(k)}\}, Z].$$

(Recall that condition (c) states that equation (10) is a numerically determinate formula for u_1.)

(c) Numerically differentiate $u_1^{(k)}$ to obtain $(du_1/dZ)^{(k)}$.

(d) Use the kth approximations which have just been computed for $u_1^{(k)}$

and $(du_1/dZ)^{(k)}$ to integrate numerically the $(k+1)$st set of approximate equations:

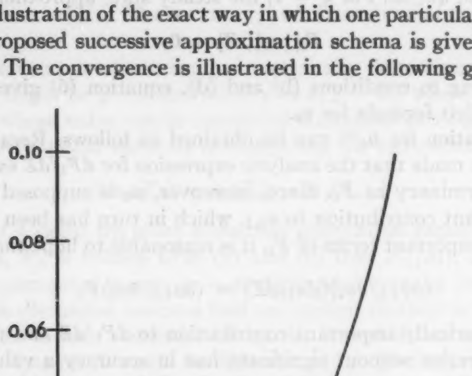
$$\text{(11a)} \qquad (du_1/dZ)^{(k)} = F_1[\{u_1^{(k)}, u_l^{(k+1)}\}, Z]$$

(an implicit algebraic equation for $u_2^{(k+1)}$)

$$\text{(11b)} \qquad (du_j/dZ)^{(k+1)} = F_j[\{u_1^{(k)}, u_l^{(k+1)}\}, Z]$$

$$(3 \le j \le M, \qquad 2 \le l \le M).$$

A detailed illustration of the exact way in which one particular set of equations falls into the proposed successive approximation schema is given in Appendix C of reference [1]. The convergence is illustrated in the following graph.



Convergence of u_2: The ordinate is the fractional difference between the second and first approximations $[u_2^{(2)} - u_2^{(1)}]/u_2^{(2)}$ for the set of four simultaneous equations which describe one hypothetical free-radical flame. The maximum fractional error is always less than thirty percent and reaches 10 percent only after u_2 has dropped to one tenthousandth its initial value. The convergence of u_1 is equally good.

4. Conclusion. If the approximation schema converges to the solution (in the case of the two equation systems studied, the schema appears to converge to the correct answer), then it possesses two noteworthy advantages: (1) it succeeds when other point-by-point procedures fail because of numerical indeterminacy of one or more of the derivatives; (2) it can be readily programmed for machine calculation. Compared with the relaxation technique which also has been found to be stable towards this type of numerical indeterminacy, successive approximation requires more calculation. However, the relaxation method requires repeated use of judgment which can become difficult if the variables in the equations are badly cross-linked. Although by far the greater part of the calculational labor in a

relaxation treatment can be programmed for machine, that part which requires judgment would be difficult to code.

The disadvantage of successive approximation lies in increasing the length of the calculation. However, as the numerical indeterminacy grows more stringent, the initial approximation improves and the number of iterations required decreases. Moreover, this is a comparatively less serious disadvantage for a method suitable for machine use. Thus the number of times the calculation is to be repeated for a different set of input data, the character of the equations with respect to cross-linking, and the comparative labor of constructing the machine programs must all be considered when deciding which method will prove most convenient for any particular study.

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1. EDWIN S. CAMPBELL, CM-847, University of Wisconsin Naval Research Laboratory, July 11, 1955.

A Smoothest Curve Approximation

A practical problem which often comes up in numerical work is the fitting of a curve to a finite set of known values in order to perform various operations such as integration. The usual method of approximation consists of fitting the points with one or more polynomials (independent of each other). By letting there be more points for each polynomial, with the polynomials being of comparable order, the error of approximation becomes asymptotic to a higher power of the interval length.

However, error analyses for such methods usually depend upon the boundedness of some derivative of a correspondingly high order [1]. But even if the function to be approximated is analytic, its correspondingly high order derivative may be of sufficient magnitude that for the given interval size, a simpler method would give better results. For instance, a fitting with an eighth order polynomial gives the following rule:

$$\int_0^8 f(t)\,dt = \frac{8}{28350}\,[989 f(0) + 5888 f(1) - 928 f(2) + 10496 f(3) - 4540 f(4) + 10496 f(5) - 928 f(6) + 5888 f(7) + 989 f(8)].$$

An application of this formula to a positive function which was everywhere small over the range, apart from a sharp peak in the center, would lead to a negative result. Try to convince a prospector that there is a negative amount of mineral on his land because he finds a rich strike in the middle of it!

Frequently one contents oneself with a simpler rule which is repeated in blocks of so many intervals per block. However, this usually introduces discontinuities in the first derivative at the junctions of the blocks. If one were to integrate an in-

terval broken up into 1000 equal subintervals, why should the 500th point be given twice as big a weighting as the 501st point (the middle point), as would be the case using Simpson's Rule? While this may not be a great objection for functions which do not undergo great changes of nature over several intervals, it may be quite unfortunate for those that do.

The main problem to be considered in this paper is the problem of obtaining and examining an integration rule based on integrating a function passing through a given set of points such that the function will have a small amount of twisting, and such that whatever twisting is necessary will be spread out. To be more specific, we will minimize

$$\int \left[\frac{d^2}{dt^2} f(t)\right]^2 dt,$$

where f ranges over an appropriate set of fitting functions, say those which are continuous and have continuous first and second derivatives.

It would be improper to consider an "angle-like" function, such as $f_0(t) = t$ $(0 \le t \le 1)$ and $f_0(t) = 1$ $(1 \le t \le 2)$, and argue that since $f_0''(t) = 0$ almost everywhere,

$$\int_0^2 [f_0''(t)]^2 dt = 0.$$

In fact if $\{f_i\}$ $(i = 1, 2, 3, \cdots)$ is a sequence of appropriate functions which converge to f_0, then

$$\lim_{i \to \infty} \int_0^2 [f_i''(t)]^2 dt = \infty.$$

Another application of this problem is that of bending a stick a little bit. A bent stick assumes a position which, subject to the bending constraints, will minimize its potential energy. Assuming the stick to be originally straight and uniform, Hooke's Law implies that its potential energy will be proportional to the integral along its arc length of the square of its curvature. If the problem is two dimensional, and if the bending is sufficiently slight that the arc length may be considered as being practically proportional to some coordinate axis, then we get the problem of minimizing

$$\int \left[\frac{d^2}{dt^2} f(t)\right]^2 dt.$$

In considering this stick application, one may also obtain the function given in Theorem 1 by mechanically balancing torques against curvatures.

1. **Theorem.** *Let $t_0 < t_i < \cdots < t_n$. For each $i = 0, \cdots, n-1$, let f be a cubic on the interval $[t_i, t_{i+1}]$. Also let*

$$\frac{d^2}{dt^2} f(t_0) \equiv f''(t_0) = 0 = f''(t_n).$$

For $i = 1, \cdots, n-1$, let $f(t_i-) = f(t_i+)$, $f'(t_i-) = f'(t_i+)$ and $f''(t_i-) = f''(t_i+)$. Let $g(\lambda)$ be any admissible function on $[t_0, t_n]$ such that $g(t_i) = f(t_i)$ for

$i = 0, \cdots, n$. Then

$$\int_{t_0}^{t_n} [g''(t)]^2\, dt > \int_{t_0}^{t_n} [f''(t)]^2\, dt$$

if and only if $g \neq f$.

Proof: Define $\eta(t)$ as $g(t) - f(t)$. Then since f''' is constant in each interval

$$\int_{t_0}^{t_n} [g''(t)]^2 dt - \int_{t_0}^{t_n} [f''(t)]^2 dt = \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \eta''(t) f''(t)\, dt$$

$$= \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2\sum_{i=0}^{n-1} [\eta'(t_{i+1}) f''(t_{i+1}) - \eta'(t_i) f''(t_i)]$$

$$- 2\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f'''(t)\eta'(t)\, dt = \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2[\eta'(t_n) f''(t_n)$$

$$- \eta'(t_0) f''(t_0)] - 2\sum_{i=0}^{n-1} f'''\ (i\text{-th interval})\ [\eta(t_{i+1}) - \eta(t_i)]$$

$$= \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2[0 - 0] - 2\sum_{i=0}^{n-1} 0$$

which is positive for admissible non-zero η's.

Note that were t_0 and t_n interior to the end points of the interval on which f is defined, then for minimizing $\int [f''(t)]^2 dt$, f would be the same as before on $[t_0, t_n]$. Also, f would be linear at the left of t_0 and the right of t_n, and the first (as well as the second) derivatives would match at t_0 and t_n.

The function f of Theorem 1 consists of n cubics, and each cubic contains four coefficients. The restriction that f passes through $f(t_i)$ at t_i gives $2n$ determining equations. The conditions $f'(t_i-) = f'(t_i+)$ and

$$f''(t_i-) = f''(t_i+) \qquad (i = 1, \cdots, n-1)$$

give $n - 1$ equations each. The equations $f''(t_0) = 0$ and $f''(t_n) = 0$ give still two more, giving a total of $4n$. Therefore we have $4n$ linear equations to determine the $4n$ coefficients of f.

2. *Definition*. In Theorem 1, let $t_i = i$ $(0 \leq i \leq n)$. Then for m an integer $(0 \leq m \leq n)$, define $w(m, n)$ as $\int_0^n f(t)dt$ where $f(i) = \delta_{m,i}$ $(i = 0, \cdots, n)$.

The following is a table of w's for $n \leq 10$. In this table $w(m, n)$ is the $m + 1$-st entry of line n. Also the common denominator is factored out at the left.

$\frac{1}{2}$ [1; 1]

$\frac{1}{8}$ [3; 10; 3]

$\frac{1}{10}$ [4; 11; 11; 4]

$\frac{1}{28}$ [11; 32; 26; 32; 11]

$\frac{1}{38}$ [15; 43; 37; 37; 43; 15]

$\frac{1}{104}$ [41; 118; 100; 106; 100; 118; 41]

$\frac{1}{142}$ [56; 161; 137; 143; 143; 137; 161; 56]

$\frac{1}{388}$ [153; 440; 374; 392; 386; 392; 374; 440; 153]

$\frac{1}{530}$ [209; 601; 511; 535; 529; 529; 535; 511; 601; 209]

$\frac{1}{1448}$ [571; 1642; 1396; 1462; 1444; 1450; 1444; 1462; 1396; 1642; 571].

Since f (and therefore $\int f dt$) is linearly dependent on the $f(t_i)$, we get the integration rule $\int_0^n f(t)dt = \sum_{m=0}^{n} f(m)w(m, n)$. For instance, from the third line of the table,

$$\int_0^3 f(t)dt = \frac{1}{10}[4f(0) + 11f(1) + 11f(2) + 4f(3)].$$

Notice that as max $[m; n - m]$ gets large, $|w(m, n) - 1|$ gets small.

The following rules which follow from Theorems 6 and 7 are probably the most useful for writing down the w's. Define $D(n)$ as the common denominator for the $w(m, n)$'s. Since

$$D(2n) = 2\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{2n} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{2n}\right]$$

and

$$D(2n+1) = \sqrt{2}\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{2n+1} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{2n+1}\right],$$

it follows that $D(2n + 2) = D(2n) + 2D(2n + 1)$ and

$$D(2n + 3) = D(2n + 1) + D(2n + 2).$$

A similar rule also holds for the numerators. For $m < 2n$, $D(2n + 2)w(m, 2n + 2) = D(2n)w(m, 2n)$

$$+ 2D(2n + 1)w(m, 2n + 1),$$

and for $m < 2n + 1$, $D(2n + 3)w(m, 2n + 3) = D(2n + 1)w(m, 2n + 1) + D(2n + 2)w(m, 2n + 2)$. For

$$0 < m < n,\ w(m, n) = 1 + (-1)^{m+1}\frac{D(|2m - n|)}{2D(n)}$$

where $D(0) = 4$. Also note the obvious fact that $w(m, n) = w(n - m, n)$.

3. ***Definition.*** For n a positive integer, consider a function f on $[0, n]$ such that f is a cubic on $[i, i + 1]$ $(i = 0, \cdots, n - 1)$; $f(i) = 0$ $(i = 0, \cdots, n)$;

$$f'(0) \neq 0; f''(n) = 0;$$

and for $i = 1, \cdots, n - 1$, $f'(i-) = f'(i+)$ and $f''(i-) = f''(i+)$. Then define $R(n)$ as $f''(0)/f'(0)$. Define $I(n)$ as $\int_0^n f(t)dt/f'(0)$. Also define $R(0) = I(0) = 0$.

Define x as the number $2 + \sqrt{3}$. Define y as $\frac{1+\sqrt{3}}{\sqrt{2}} = \sqrt{x}$.

4. Lemma. *For n a non-negative integer,* $R(n) = -2\sqrt{3}\frac{x^n - x^{-n}}{x^n + x^{-n}}$.

Proof: $R(0) = 0 = -2\sqrt{3}\frac{x^0 - x^{-0}}{x^0 + x^{-0}}$. To use mathematical induction, let us show that the validity of this formula for $R(n)$ implies its validity for $R(n + 1)$.

Let f be the function of Definition 3 defined on $[0, n+1]$. $f(0) = f(1) = 0$ and $f'(0) \neq 0$ imply that there exist real numbers a and $\lambda \neq 0$ such that $f(t) = \lambda[t - at^2 + (a-1)t^3]$ for $0 \leq t \leq 1$. Then

$$R(n) = \frac{f''(1)}{f'(1)} = \frac{4a-6}{a-2}$$

implies that $a = \dfrac{6 - 2R(n)}{4 - R(n)}$. Therefore

$$R(n+1) = \frac{f''(0)}{f'(0)} = -2a = -4\left[\frac{3 - R(n)}{4 - R(n)}\right]$$

which, substituting for $R(n)$, equals

$$-2\sqrt{3}\,\frac{x^{n+1} - x^{-(n+1)}}{x^{n+1} + x^{-(n+1)}}.$$

5. **Lemma.** *For n a non-negative integer,* $I(n) = \dfrac{1}{12} - \dfrac{(-1)^n}{6(x^n + x^{-n})}$;

Proof. $I(0) = 0 = \dfrac{1}{12} - \dfrac{(-1)^0}{6(x^0 + x^{-0})}$. To use mathematical induction, let us show that the validity of this formula for $I(n)$ implies its validity for $I(n+1)$. Let f, a, and λ be the same as in the proof of Lemma 4. Then

$$I(n+1) = \frac{1}{f'(0)}\int_0^{n+1} f(t)dt = \frac{1}{\lambda}\left[\int_0^1 f(t)dt + f'(1)I(n)\right]$$

$$= \frac{1}{4} - \frac{a}{12} + (a-2)I(n) = \frac{1}{4} - \frac{R(n+1)}{2}\left[I(n) - \frac{1}{12}\right] - 2I(n)$$

$$= \frac{1}{12} - \frac{(-1)^{n+1}}{6[x^{n+1} + x^{-(n+1)}]}$$

on substituting for $R(n+1)$ and $I(n)$.

6. **Theorem.** *For n a positive integer*

$$w(0, n) = \frac{y^{n+1} - (-y)^{-(n+1)}}{2\sqrt{6}[y^n + (-y)^{-n}]}.$$

Proof. Let f be the f of Definition 2. Then $f(0) = 1$, $f(1) = 0$ and $f''(0) = 0$ imply that there exists a real number a such that

$f(t) = 1 - at + (a-1)t^3$ for $0 \leq t \leq 1$.

$R(n-1) = \dfrac{f''(1)}{f'(1)} = \dfrac{6(a-1)}{2a-3}$. Therefore $a = \dfrac{3R(n-1) - 6}{2R(n-1) - 6}$. Using

$$R(n) = -4\,\frac{3 - R(n-1)}{4 - R(n-1)}$$

from Lemma 4, we see that $R(n-1) = 4\dfrac{3+R(n)}{4+R(n)}$ and so $a = 3 + \dfrac{6}{R(n)}$.

$$w(0, n) = \int_0^n f(t)dt = \int_0^1 f(t)dt + y'(1)I(n-1)$$
$$= \frac{3}{4} - \frac{a}{4} + [2a-3]\left[\frac{1}{12} - \frac{(-1)^{n-1}}{6(x^{n-1}+x^{1-n})}\right]$$
$$= \frac{1}{2} - \frac{a}{12} + \frac{2a-3}{6}\frac{(-1)^n}{x^{n-1}+x^{1-n}}$$
$$= \frac{1}{4} - \frac{1}{2R(n)} + \left[\frac{1}{2} + \frac{2}{R(n)}\right]\frac{(-1)^n}{x^{n-1}+x^{1-n}}.$$

On substituting for $R(n)$ from lemma 4 and after some manipulation, we obtain

$$w(0, n) = \frac{y^{n+1} - (-y)^{-(n+1)}}{2\sqrt{6}[y^n + (-y)^{-n}]}.$$

7. THEOREM. *For*

$$0 < m < n,\ w(m, n) = 1 - (-1)^m \frac{y^{n-2m} + (-y)^{2m-n}}{2[y^n + (-y)^{-n}]}.$$

Proof. Let f be the function of Theorem 1 where $t_i = i - m$ $(i = 0, \cdots, n)$ and $f(t_i) = \delta_{m,i}$. Then $f(0) = 1, f(-1) = f(+1) = 0, f'(0-) = f'(0+)$ and $f''(0-) = f''(0+)$ imply there exist real numbers a and b so that $f(t) = 1 + at + bt^2 - (1+a+b)t^3$ for $0 \le t \le 1$ and $f(t) = 1 + at + bt^2 + (1-a+b)t^3$ for $-1 \le t \le 0$. Define $\alpha = n - m - 1$ and $\beta = m - 1$. Then

$$R(\alpha) = \frac{f''(1)}{f'(1)} = \frac{6+6a+4b}{3+2a+b}$$

and

$$R(\beta) = -\frac{f''(-1)}{f'(-1)} = \frac{6-6a+4b}{3-2a+b}.$$

Solving these last two equations, we obtain

$$a = \frac{3R(\alpha) - 3R(\beta)}{2R(\alpha)R(\beta) - 7R(\alpha) - 7R(\beta) + 24}$$

and

$$b + 3 = \frac{-6R(\alpha) - 6R(\beta) + 36}{2R(\alpha)R(\beta) - 7R(\alpha) - 7R(\beta) + 24}.$$

Now,

$$w(m, n) - 1 = \int_{-m}^{n-m} f(t)dt - 1$$
$$= I(\alpha)f'(1) - I(\beta)f'(-1) + \int_{-1}^{1} f(t)dt - 1$$
$$= I(\alpha)[-2a - (b+3)] + I(\beta)[2a - (b+3)] + \frac{1}{6}(b+3).$$

Substituting the values for a and $b+3$ into the last equation and multiplying both sides by the common denominator of a and $b+3$, we get

$$[2R(\alpha)R(\beta) - 7R(\alpha) - 7R(\beta) + 24][w(m, n) - 1]$$
$$= 12I(\alpha)[R(\beta) - 3] + 12I(\beta)[R(\alpha) - 3] - R(\alpha) - R(\beta) + 6.$$

Now substitute for $R(\alpha)$, $R(\beta)$, $I(\alpha)$ and $I(\beta)$ the expressions of lemmas 4 and 5 and multiply both sides of the equation by $(x^{\alpha} + x^{-\alpha})\ (x^{\beta} + x^{-\beta})$.

Then collect like terms to get

$$[(48 + 28\sqrt{3})x^{\alpha+\beta} + (48 - 28\sqrt{3})x^{-(\alpha+\beta)}]\,[w(m, n) - 1]$$
$$= (6 + 4\sqrt{3})\ (-1)^{\beta}x^{\alpha} + (6 + 4\sqrt{3})\ (-1)^{\alpha}x^{\beta}$$
$$+ (6 - 4\sqrt{3})\ (-1)^{\beta}x^{\alpha} + (6 - 4\sqrt{3})\ (-1)^{\beta}x^{-\alpha}.$$

Divide both sides of this equation by $2\sqrt{3}$ and substitute $x = 2 + \sqrt{3}$ to get

$$2[x^{\alpha+\beta+2} - x^{-(\alpha+\beta+2)}]\,[w(m, n) - 1]$$
$$= (-1)^{\beta}x^{\alpha+1} + (-1)^{\alpha}x^{\beta+1} - (-1)^{\beta}x^{-(\alpha+1)} - (-1)^{\alpha}x^{-(\beta+1)}.$$

Since $x = y^2$, we may divide both sides of this equation by

$$[y^{\alpha+\beta+2} - (-y)^{-(\alpha+\beta+2)}]\,[y^{\alpha+\beta+2} + (-y)^{-(\alpha+\beta+2)}]$$

which using the binomial expansion, equals

$$\sum_{j=2}^{k+1} \frac{2^{j}k!B_j}{j!(k-j+1)!n^j} + \frac{y}{n\sqrt{6}}\,\frac{1 - (-x)^{-(n+1)}}{1 + (-x)^{-n}}$$
$$- \frac{1}{n}\sum_{i=0}^{k}\sum_{m=0}^{n} \frac{n^{-i}\,(-2m)^{i}k!}{i!(k-i)!}\,\frac{(-x)^{-m}}{1 + (-x)^{-n}}$$
$$= \sum_{i=1}^{k} \frac{2^{i+1}k!}{(i+1)!(k-i)!n^{i+1}}\left[B_{i+1} - \frac{i+1}{2[1 + (-x)^{-n}]}\sum_{m=0}^{n}(-m)^{i}(-x)^{-m}\right]$$
$$= \frac{1}{2}\sum_{i\geq 1} \frac{h^{i+1}}{(i+1)!}\left[\frac{d^i}{dt^i}P(1) + (-1)^{i}\frac{d^i}{dt^i}P(-1)\right]$$
$$\left[B_{i+1} - \frac{i+1}{2[1 + (-x)^{-n}]}\sum_{m=0}^{n}(-m)^{i}(-x)^{-m}\right].$$

8. *Definition.* For $i - 1$ a positive integer, define B_i as i factorial times the coefficient of t^i in the power series expansion of $t/(e^t - 1)$. In other words, let $0 = B_3 = B_5 = B_7 = \cdots$, and let $B_2, B_4, B_6, B_8, \cdots$ be the Bernoulli's Numbers

$$\frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}, -\frac{691}{2730}, \frac{7}{6}, \ldots.$$

9. Theorem. *Let $P(t)$ be a finite polynomial on the interval (a, b). Then*

$$\int_a^b P(t)dt = \sum_{m=0}^{n} P(a + mh)w(m, n)\,h$$
$$- \sum_{i\geq 1} \frac{h^{i+1}}{(i+1)!}\left[\frac{d^i}{dt^i}P(b) + (-1)^{i}\frac{d^i}{dt^i}P(a)\right]$$
$$\left[B_{i+1} - \frac{i+1}{2[1+(-x)^{-n}]}\sum_{m=0}^{n}(-m)^{i}(-x)^{-m}\right]$$

where h is the sub-interval length $= \dfrac{b - a}{n}$.

Proof: If this theorem holds for a given $a < b$, then for λ a positive real number, this theorem also holds for end points λa and λb, since new $h = \lambda$ old h and

$\frac{d^i}{dt^i} P(t)$ at $t = a$ (or b) $= \lambda^{-i} \frac{d^i}{dt^i} P\left(\frac{t}{\lambda}\right)$ at $t = \lambda a$ (or λb). Also, the three terms of

this equation are invariant under translations. Therefore, it is sufficient to prove this theorem for $-a = b = 1$. Also, since this equation is linear in P, it is sufficient to verify it for $P(t) = t^k$ where k is a non-negative integer.

If k is odd, then $P(t) = -P(-t)$ implies that all three terms of the equation of this theorem are zero. Therefore, we may let k be even.

$$\frac{1}{2}\left[\sum_{m=0}^{n} P(-1 + mh)\, w(m, n)\, h - \int_{-1}^{1} P(t)dt\right]$$

$$= \frac{1}{n}\left[\sum_{m=0}^{n}\left(\frac{2m - n}{n}\right)^k w(m, n)\right] - \frac{1}{k+1}$$

$$= \frac{1}{n}\left\{\sum_{m=1}^{n-1}\left(\frac{n - 2m}{n}\right)^k\left[1 - (-1)^m \frac{y^{n-2m} + (-y)^{2m-n}}{2[y^n + (-y)^{-n}]}\right] + 2w(0, n)\right\} - \frac{1}{k+1}$$

$$= \frac{1}{n}\left\{\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k\left[1 - (-1)^m \frac{x^{-m}}{1 + (-x)^{-n}}\right] - 1 + 2w(0, n)\right\} - \frac{1}{k+1}.$$

If n is even, $\frac{1}{n}\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k = \frac{2}{n}\sum_{i=0}^{n/2}\left(\frac{2i}{n}\right)^k$.

If n is odd, $\frac{1}{n}\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k = \frac{2}{n}\sum_{i=0}^{(n-1)/2}\left(\frac{2i + 1}{n}\right)^k$.

Using the Euler-Maclaurin sum formula [2], either of these last two expressions equals

$$\frac{1}{k+1} + \frac{1}{n} + \sum_{j=2}^{k+1} \frac{2^j k! B_j}{j!(k - j + 1)!\, n^j}.$$

Substituting, we get

$$\frac{1}{2}\left[\sum Pwh - \int Pdt\right] = \sum_{j=2}^{k+1} \frac{2^j k! B_j}{j!(k - j + 1)!\, n^j} + \frac{2w(0, n)}{n} - \frac{1}{n}\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k \frac{(-x)^{-m}}{1 + (-x)^{-n}}$$

and

$$D(2n + 1) = \sqrt{2}[y^{2n+1} + (-y)^{-(2n+1)}],$$

we see that

$$\phi_1\left(\frac{1}{x}\right) = -\frac{1}{6}.$$

$$\phi_2\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{36}[-D(1)] = -\frac{\sqrt{3}}{18}.$$

$$\phi_3\left(\frac{1}{x}\right) = \frac{1}{36}\left[4 - \frac{1}{2}D(2)\right] = 0.$$

$$\phi_4\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{216}[11D(1) - D(3)] = \frac{\sqrt{3}}{18}.$$

$$\phi_5\left(\frac{1}{x}\right) = \frac{1}{216}\left\{-66 + \frac{1}{2}[26D(2) - D(4)]\right\} = \frac{1}{9}.$$

$$\phi_6\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{1296}[-302D(1) + 57D(3) - D(5)] = -\frac{\sqrt{3}}{18}.$$

$$\phi_7\left(\frac{1}{x}\right) = \frac{1}{1296}\left\{2416 + \frac{1}{2}[-1191D(2) + 120D(4) - D(6)]\right\} = -\frac{5}{9}.$$

$$\phi_8\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{7776}[15{,}619D(1) - 4293D(3) + 247D(5) - D(7)] = -\frac{17\sqrt{3}}{54}.$$

$$\phi_9\left(\frac{1}{x}\right) = \frac{1}{7776}\left\{-156{,}190 + \frac{1}{2}[88{,}234D(2) - 14{,}608D(4) + 502D(6) - D(8)]\right\} = \frac{7}{3}.$$

Therefore, the equation

$$\int_a^b P(t)dt = T(n) + \sum_{m=0}^{n} P(a+mh)w(m,n)h - \sum_{i\geq 1}\frac{h^{i+1}}{(i+1)!}\left[\frac{d^i}{dt^i}P(b) + (-1)^i\frac{d^i}{dt^i}P(a)\right]\left[B_{i+1} - (-1)^i\frac{i+1}{2}\phi_i\left(\frac{1}{x}\right)\right].$$

gives us the series of this theorem.

10. Corollary. $\sum_{m=0}^{n} w(m,n) = n.$

11. Theorem. *Let $P(t)$ be a finite polynomial on the interval (a, b). Then*

$$\begin{aligned}\int_a^b P(t)dt &= T(n) + \sum_{m=0}^{n} P(a+mh)w(m,n)h \\ &\quad - \frac{\sqrt{3}}{72}h^3[P^{\text{ii}}(b) + P^{\text{ii}}(a)] + \frac{1}{720}h^4[P^{\text{iii}}(b) - P^{\text{iii}}(a)] \\ &\quad + \frac{\sqrt{3}}{864}h^5[P^{\text{iv}}(b) + P^{\text{iv}}(a)] - \frac{1}{2016}h^6[P^{\text{v}}(b) - P^{\text{v}}(a)] \\ &\quad - \frac{\sqrt{3}}{25{,}920}h^7[P^{\text{vi}}(b) + P^{\text{vi}}(a)] + \frac{29}{518{,}400}h^8[P^{\text{vii}}(b) - P^{\text{vii}}(a)] \\ &\quad - \frac{17\sqrt{3}}{4{,}354{,}560}h^9[P^{\text{viii}}(b) + P^{\text{viii}}(a)] - \frac{31}{9{,}580{,}032}h^{10}[P^{\text{ix}}(b) - P^{\text{ix}}(a)] \\ &\quad + \cdots, \textit{ where } h = \frac{b-a}{n} \textit{ and } \lim_{n\to\infty}(2+\sqrt{3})^n T(n) = 0.\end{aligned}$$

Proof: For each i, $\sum_{m=0}^{\infty}(-m)^i(-x)^{-m}$ is an alternating series. Therefore, for sufficiently large n,

$$\left|\sum_{m=0}^{\infty}(-m)^i(-x)^{-m} - \sum_{m=0}^{n}(-m)^i(-x)^{-m}\right| < (n+1)^i x^{-(n+1)}.$$

Since h^{i+1} is proportional to $1/n^{i+1}$,

$$\lim_{n\to\infty} x^n\left\{\int_a^b P(t)dt - \sum_{m=0}^{n} P(a+mh)w(m,n)h \right.$$
$$+ \sum_{i\geq 1}\frac{h^{i+1}}{(i+1)!}\left[\frac{d^i}{dt^i}P(b) + (-1)^i\frac{d^i}{dt^i}P(a)\right]$$
$$\left.\left[B_{i+1} - \frac{i+1}{2}\sum_{m=0}^{\infty}(-m)^i(-x)^{-m}\right]\right\} = 0.$$

If we define $\phi_i(t$ as) $\sum_{m=0}^{\infty} m^i(-t)^m$, then $\phi_{i+1}(t) = \frac{d}{dt}\phi_i(t)$.

Since $\phi_0(t) = \dfrac{1}{1+t}$, $\phi_1(t) = \dfrac{-t}{(1+t)^2}$; $\phi_2(t) = \dfrac{-t+t^2}{(1+t)^3}$;

$$\phi_3(t) = \frac{-t+4t^2-t^3}{(1+t)^4};\ \phi_4(t) = \frac{-t+11t^2-11t^3+t^4}{(1+t)^5};$$

$$\phi_5(t) = \frac{-t+26t^2-66t^3+26t^4-t^5}{(1+t)^6};$$

$$\phi_6(t) = \frac{-t+57t^2-302t^3+302t^4-57t^5+t^6}{(1+t)^7}$$

etc., and keeping in mind that

$$1+\frac{1}{x} = \frac{\sqrt{6}}{y} = \sqrt{\frac{6}{x}},\ D(2n) = 2\left[y^{2n} + (-y)^{-2n}\right]$$

to get

$$w(m,n) - 1 = -\frac{y^{\alpha+1}(-y)^{-(\beta+1)} + y^{\beta+1}(-y)^{-(\alpha+1)}}{2\left[y^{\alpha+\beta+2} + (-y)^{-(\alpha+\beta+2)}\right]}$$
$$= (-1)^m\frac{y^{n-2m} + (-y)^{2m-n}}{2\left[y^n + (-y)^{-n}\right]}.$$

12. *Examples*. Let $g(t)$ be a function on $[-1, 1]$ and let us consider the error $\sum_{m=0}^{n} g\left(-1+\frac{2m}{n}\right)\frac{2\,w(m,n)}{n} - \int_{-1}^{1} g(t)\ dt$. If g is constant or $g(t) = -g(-t)$, then the error is zero, as is true of most rules.

If we let $n = 10$ and consider $g(t)$ as t^2, t^4 and t^6 respectively, then the error is 7.7×10^{-4}, 4.5×10^{-3}, and 1.1×10^{-2} respectively. If we now apply the end point terms of Theorem 11, the remaining errors are -2.9×10^{-9}, -1.8×10^{-8} and -4.2×10^{-8} respectively. If we let $n = 14$, then the above quoted errors become 2.8×10^{-4}, 1.6×10^{-3}, and 4.0×10^{-3} and also -5.5×10^{-12}, -3.2×10^{-11} and -8.1×10^{-11}. The corresponding errors using Simpson's Rule are 0, 4.3×10^{-4} and 2.1×10^{-3} for 10 intervals and 0, 1.1×10^{-4} and 5.5×10^{-4} for 14 intervals.

If we let $g(t) = \cos(\pi t/4)$ and $n = 12$, then the error is -1.7×10^{-5}. If we apply the $\frac{1}{720} h^4 [g'''(1) - g'''(-1)]$ correction term, the remaining error is -7.1×10^{-7}. The corresponding error using Simpson's Rule is 6.7×10^{-6}.

13. *Semi-Infinite Interval.* Let us define $w(m, \infty)$ as $\lim_{n\to\infty} w(m, n)$. Then

$$w(0, \infty) = \frac{1}{4} + \frac{\sqrt{3}}{12}$$

and

$$w(m, \infty) = 1 - \frac{1}{2}(-x)^{-m} \text{ for } m > 0.$$

Given a function $g(t)$ for which the integral $\int_0^\infty g(t)dt$ is convergent, one may approximate this integral by the expression $h \sum_{m=0}^{N} w(m, \infty)g(mh)$ where N is sufficiently large that $h \sum_{m=N+1}^{\infty} w(m, \infty)g(mh)$ is insignificant. If one also knows the first few derivatives of $g(t)$ at $t = 0$, one may also add the terms obtained from Theorem 11.

If $g(t) = e^{-\lambda t}$, then the series

$$h \sum_{m=0}^{\infty} w(m, \infty)g(m,h) - \frac{\sqrt{3}}{72} h^3 g''(0) - \frac{1}{720} h^4 g'''(0)$$
$$+ \frac{\sqrt{3}}{864} h^5 g''''(0) + \frac{1}{2016} h^6 g'''''(0) + \cdots$$

will converge to $\int_0^\infty g(t)\,dt$ whenever $h\lambda < 2\pi$.

For some functions such as $g(t) = e^{-t^2}$, this sequence of derivatives is divergent for any $h > 0$. However, this fact does not imply that using this formula with the first few derivative terms would not do a good job of approximation.

Notice that the weight terms $w(m, \infty)$ are easy to compute on a machine, requiring only one multiplication per term.

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1. J. B. SCARBOROUGH, *Numerical Mathematical Analysis*, Second Edition, Oxford Univ. Press, London, 1950, Chapter VIII.
2. F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York, 956. p. 150.

Numerical Integration over the Spherical Shell

1. Introduction. Numerical integration over three-dimensional regions has received relatively little attention. The papers listed in the bibliography give the work on this subject known to the author. With regards the sphere, Hammer and Wymore [6], have given formulas with specific polynomial accuracy. This paper gives a formula for arbitrarily high polynomial accuracy and generalizes the region to the spherical shell.

2. Type of formula. The type of formula considered is a weighted sum of integrand values

$$\iiint f(x, y, z)dxdydz \doteq \sum_{i=1}^{n} a_i f(x_i, y_i, z_i),$$

where the dot over the equals sign is meant to signify that in general the right hand side will only be an approximation to the left hand side. The region under consideration is the spherical shell of inner radius R and outer radius one. By a theorem of Hammer and Wymore [7], a formula over any region which is an affine transformation of this spherical shell can be easily obtained. The criterion used to determine the weights, a_i, and the points, (x_i, y_i, z_i), is that the formula shall be exact whenever the integrand function is a polynomial in x, y, and z of degree at most s.

A numerical integration formula, whether it be in one, two, or more variables, is said to have accuracy s provided the formula is exact whenever the integrand function is a polynomial of degree s or less, and provided there exists at least one polynomial of degree $s + 1$ for which the formula is not exact.

3. Numerical integration over the spherical shell. We first transform the problem to one in spherical coordinates. Under the transformation

$$x = r \sin \phi \cos \theta$$
$$y = r \sin \phi \sin \theta$$
$$z = r \cos \phi$$

the integral $\iiint f(x, y, z)dxdydz$ over the spherical shell with inner radius R and outer radius one can be written as

$$\int_R^1 \int_0^\pi \int_0^{2\pi} r^2 \sin \phi F(\theta, \phi, r)d\theta d\phi dr,$$

where $F(\theta, \phi, r) \equiv f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$. Since any formula for the integral in rectangular form is to be exact whenever the integrand function is a polynomial of degree at most s in x, y, and z, any formula for the integral in spherical form must be exact whenever $F(\theta, \phi, r)$ is a polynomial of degree at most s in $r \sin \phi \cos \theta$, $r \sin \phi \sin \theta$, and $r \cos \phi$. The type of formula we attempt to find is

$$(1) \quad I = \int_R^1 \int_0^\pi \int_0^{2\pi} r^2 \sin \phi F(\theta, \phi, r)d\theta d\phi dr \doteq \sum_i \sum_j \sum_k D_{ijk} F(\theta_i, \phi_j, r_k),$$

which is a weighted sum of integrand values where the limits on i, j, and k are to be determined. We now state the main result of this paper in the following theorem.

THEOREM. *If it be required that*

(a) *formula* (1) *have accuracy* $s = 4m + 3$, $m = 0, 1, 2, \cdots$, *in* $r \sin \phi \cos \theta$, $r \sin \phi \sin \theta$, *and* $r \cos \phi$,
(b) *the evaluation points be taken at all intersection points of the planes* $\theta = \theta_i$, *the planes* $\phi = \phi_j$, *and the spheres* $r = r_k$,
(c) *the evaluation points under this configuration be minimum in number,*

then it is both necessary and sufficient for the existence of a unique set of real weights D_{ijk} *that*

(a) $\theta_i = 2\pi_i/(s + 1)$, $i = 1, 2, \cdots, s + 1$,
(b) *the* $\cos \phi_j$ *be the* $2m + 2$ *zeros of the Legendre polynomial* P_{2m+2} *of degree* $2m + 2$, *orthogonalized on* $[-1, 1]$,
(c) *the* r_k^2 *be the* $m + 1$ *zeros of the polynomial in* r^2 *of degree* $m + 1$, $Q_{m+1}(r^2)$, *where* $\int_R^1 rQ_{m+1}(r^2)T_m(r^2)dr^2 = 2\int_R^1 r^2Q_{m+1}(r^2)T_m(r^2)dr = 0$, *and where* $T_m(r^2)$ *is an arbitrary polynomial in* r^2 *of degree* m *or less.*

The unique weights D_{ijk} are equal to $A_iB_jC_k$ where

$$A_i = \frac{2\pi}{s + 1}, \quad i = 1, 2, \cdots, s + 1,$$

$$B_j = \frac{1}{P'_{2m+2}(y_j)} \int_{-1}^{1} \frac{P_{2m+2}(y)}{y - y_j}\, dy, \; j = 1, 2, \cdots, 2m + 2,$$

$$C_k = \frac{1}{Q'_{m+1}(r_k^2)} \int_R^1 \frac{r^2 Q_{m+1}(r^2)}{r^2 - r_k^2}\, dr, \; k = 1, 2, \cdots, m + 1,$$

or

$$C_k = \frac{1}{2Q'_{m+1}(t_k)} \int_{R^2}^{1} \frac{\sqrt{t}\, Q_{m+1}(t)}{t - t_k}\, dt, \text{ where } t = r^2.$$

In the first expression for C_k, the notation $Q'_{m+1}(r_k^2)$ indicates a derivative with respect to r^2.

Proof. Because of the arrangement of the points we have chosen, we write the left hand part of (1) as

$$I = \int_R^1 r^2 \int_0^\pi \sin \phi \int_0^{2\pi} F(\theta, \phi, r) d\theta d\phi dr.$$

Now define

$$H(\phi, r) = \int_0^{2\pi} F(\theta, \phi, r) d\theta,$$

$$G(r) = \int_0^\pi \sin \phi H(\phi, r) d\phi,$$

which gives

$$I = \int_R^1 r^2 G(r) dr.$$

We now look for three formulas, namely

$$(2)\qquad H(\phi, r) = \int_0^{2\pi} F(\theta, \phi, r)d\theta \doteq \sum_{i=1}^{p} A_i F(\theta_i, \phi, r),$$

$$(3)\qquad G(r) = \int_0^{\pi} \sin\phi H(\phi, r)d\phi \doteq \sum_{j=1}^{q} B_j H(\phi_j, r),$$

$$(4)\qquad I = \int_R^1 r^2 G(r)dr \doteq \sum_{k=1}^{v} C_k G(r_k).$$

When these three formulas have been obtained, we will combine them to obtain formula (1) with $D_{ijk} = A_i B_j C_k$.

Formula (2)

$$H(\phi, r) = \int_0^{2\pi} F(\theta, \phi, r)d\theta \doteq \sum_{i=1}^{p} A_i F(\theta_i, \phi, r).$$

The number p is to be determined, and this formula must be exact whenever $F(\theta, \phi, r)$ is a polynomial of degree s in $r \sin\phi \cos\theta$, $r \sin\phi \sin\theta$, and $r \cos\phi$. Since r and ϕ are parameters here, we can require formula (2) to be exact whenever F is a polynomial of degree s in $\cos\theta$ and $\sin\theta$, and hence whenever $F = 1$, $\cos\theta$, $\sin\theta$, $\cos 2\theta$, $\sin 2\theta$, $\cdots$, $\cos s\theta$, $\sin s\theta$. This leads to a system of $2s + 1$ equations

$$A_1 + A_2 + \cdots + A_p = \int_0^{2\pi} 1 d\theta = 2\pi$$

$$A_1 \cos t\theta_1 + A_2 \cos t\theta_2 + \cdots + A_p \cos t\theta_p = \int_0^{2\pi} \cos t\theta d\theta = 0$$

$$A_1 \sin t\theta_1 + A_2 \sin t\theta_2 + \cdots + A_p \sin t\theta_p = \int_0^{2\pi} \sin t\theta d\theta = 0$$

for $t = 1, 2, \cdots, s$.

One solution to this system (a sufficient condition that formula (2) have accuracy s with minimum p) is given by

$$p = s + 1$$

$$A_i = \frac{2\pi}{s+1}, \qquad i = 1, 2, \cdots, s + 1$$

$$\theta_i = \frac{2\pi i}{s+1}, \qquad i = 1, 2, \cdots, s + 1.$$

The requirement that the A_i be real leads to the conclusion that this is the only solution up to a rotation. (For the idea which led to this result, I am grateful to Professor Fritz Herzog.) The above is therefore a necessary and sufficient condition that formula (2) have accuracy s in $r \sin\phi \cos\theta$, $r \sin\phi \sin\theta$, and $r \cos\phi$.

Formula (3)

$$G(r) = \int_0^{\pi} \sin\phi H(\phi, r)d\phi \doteq \sum_{j=1}^{q} B_j H(\phi_j, r)$$

The number q has to be determined, and this formula, like (2), must be exact whenever $F(\theta, \phi, r)$ is a polynomial of degree s in $r \sin\phi \cos\theta$, $r \sin\phi \sin\theta$, and $r \cos\phi$. Considering the definition of $H(\phi, r)$, formula (3) must therefore be exact whenever $H(\phi, r)$ is a polynomial of degree s in $\cos\phi$. This leads to a system of

$s + 1$ equations

$$B_1 \cos^t \phi_1 + B_2 \cos^t \phi_2 + \cdots + B_q \cos^t \phi_q = \int_0^\pi \sin\phi \cos^t \phi d\phi$$

for $t = 0, 1, 2, \cdots, s$. Putting $\cos\phi = y$, these equations become

$$B_1 y_1^t + B_2 y_2^t + \cdots + B_q y_q^t = \int_{-1}^1 y^t dy.$$

The unique solution to this system for minimum q has been given by Gauss [4] and is

$$q = \frac{s+1}{2} = 2m + 2,$$

$y_j = \cos\phi_j =$ the $q = 2m + 2$ zeros of the Legendre polynomial P_{2m+2} of degree $2m + 2$, orthogonalized on $[-1, 1]$.

$$B_j = \frac{1}{P'_{2m+2}(y_j)} \int_{-1}^1 \frac{P_{2m+2}(y)}{y - y_j} dy,$$

where P' means the derivative of P with respect to y.

Formula (4)

$$I = \int_R^1 r^2 G(r) dr \doteq \sum_{k=1}^{v} C_k G(r_k).$$

Again, the number v has to be determined, and this formula, like (2) and (3), must be exact whenever $F(\theta, \phi, r)$ is a polynomial of degree s in $r \sin\phi \cos\theta$, $r \sin\phi \sin\theta$, and $r \cos\phi$. From the definition of $G(r)$, formula (4) must be exact whenever $G(r)$ is a polynomial in r^2 of degree $(s - 1)/2 = 2m + 1$. Putting $g(r^2) = G(r)$, formula (4) can be written

$$2I = \int_R^1 r g(r^2) dr^2 \doteq \sum_{k=1}^{v} 2C_k g(r_k^2).$$

The unique solution for the C_k and the r_k for minimum v is given by Szegö [11] and is

$$v = m + 1,$$

$r_k^2 =$ the $m + 1$ zeros of the polynomial in r^2 of degree $m + 1$, $Q_{m+1}(r^2)$, where $\int_R^1 r Q_{m+1}(r^2) T_m(r^2) dr^2 = 2 \int_R^1 r^2 Q_{m+1}(r^2) T_m(r^2) dr = 0$, and where $T_m(r^2)$ is an arbitrary polynomial in r^2 of degree m or less,

$$C_k = \frac{1}{Q'_{m+1}(r_k^2)} \int_R^1 \frac{r^2 Q_{m+1}(r^2)}{r^2 - r_k^2} dr,$$

where Q' means the derivative of Q with respect to r^2. This completes the proof of the theorem.

Several corollaries are immediate.

COROLLARY. *The weights $D_{ijk} = A_i B_j C_k$ are positive, and their sum is the volume of the spherical shell.*

COROLLARY. *All evaluation points of the formula lie between the inner and outer sphere of the spherical shell.*

COROLLARY. *The number of evaluation points is $8(m + 1)^3$.*

Although $8(m + 1)^3$ is the minimum number of points for the point configuration we have chosen, it is not necessarily the minimum number for other point configurations. For example, the formula developed by Hammer and Wymore [7] for 7th degree accuracy utilizes only 27 points, whereas the formula developed here uses 64 points for the same accuracy. The formulas presented here, however, give formulas of arbitrarily high accuracy. (By removing the restriction that s be of the form $4m + 3$, which can be done, it is possible in some cases to reduce the number of points below the number specified by $8(m + 1)^3$.

4. The weights C_k and the zeros of $Q_{m+1}(r^2)$. We list below, for $m = 0$ and for $m = 1$, and for four values of the inner radius R, the polynomials $Q_{m+1}(r^2)$, their zeros, and the weights C_k. (Decimal numbers are truncated, not rounded.)

$m = 0$

$R = 0$ $\qquad Q_1(r^2) = r^2 - \dfrac{3}{5}$

$$r_1^2 = .60000\ 00000 \qquad C_1 = .33333\ 33333$$

$R = \frac{1}{4}$ $\qquad Q_1(r^2) = r^2 - \dfrac{341}{560}$

$$r_1^2 = .60892\ 85714 \qquad C_1 = .32812\ 50000$$

$R = \frac{1}{2}$ $\qquad Q_1(r^2) = r^2 - \dfrac{93}{140}$

$$r_1^2 = .66428\ 5714 \qquad C_1 = .29166\ 66666$$

$R = \frac{3}{4}$ $\qquad Q_1(r^2) = r^2 - \dfrac{2343}{2960}$

$$r_1^2 = .79155\ 40540 \qquad C_1 = .19270\ 83333$$

$m = 1$

$R = 0$ $\qquad Q_2(r^2) = r^4 - \dfrac{10}{9} r^2 + \dfrac{5}{21}$

$$r_1^2 = .28994\ 91979 \qquad C_1 = .13877\ 79991$$
$$r_2^2 = .82116\ 19131 \qquad C_1 = .19455\ 53342$$

$R = \frac{1}{4}$ $\qquad Q_2(r^2) = r^4 - \dfrac{107{,}605}{94{,}472} r^2 + \dfrac{5{,}464{,}615}{21{,}161{,}728}$

$$r_1^2 = .31239\ 3379 \qquad C_1 = .13890\ 8126$$
$$r_2^2 = .82662\ 1355 \qquad C_2 = .18921\ 6873$$

$R = \frac{1}{2}$ $\qquad Q_2(r^2) = r^4 - \dfrac{5905}{4599} r^2 + \dfrac{63{,}005}{171{,}696}$

$$r_1^2 = .42940\ 5421 \qquad C_1 = .13053\ 6459$$
$$r_2^2 = .85456\ 9355 \qquad C_2 = .16113\ 0205$$

$R = \frac{3}{4}$ $\qquad Q_2(r^2) = r^4 - \dfrac{2{,}046{,}079{,}840}{1{,}302{,}513{,}408} r^2 + \dfrac{782{,}901{,}015}{1{,}302{,}513{,}408}$

$$r_1^2 = .65958\ 19 \qquad C_1 = .09166\ 95$$
$$r_2^2 = .91128\ 85 \qquad C_2 = .10103\ 87$$

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On a Generalization of the Prime Pair Problem

1. Introduction. One of the many unsolved problems in the theory of prime numbers concerns the celebrated conjecture by A. de Polignac [1] about primes which differ by two. Prime pairs (or twin primes) are generally believed to constitute an infinite set but to date no one has succeeded in proving or disproving this conjecture.

It was established by Brun [2] that the sum of the reciprocals of all prime pairs is bounded but whether this boundedness is due to the finiteness of the set or to the "thinness" of an infinite set has not been determined.

2. Prime n-tuples. In generalizing from prime pairs to prime n-tuples we must first define what is meant by a prime n-tuple.

Definition: a prime n-tuple is a set of n odd primes $p_1, p_2, \cdots p_n$ such that the difference $\Delta n = p_n - p_1$ is a non-trivial minimum.

The meaning of "non-trivial" will now be explained. For $n = 3$ we can exhibit a set of three consecutive odd primes such that $\Delta 3 = 4$ (i.e., 3, 5, 7). We consider this a trivial minimum, however, because (with one exception) in every set of three consecutive odd integers one of the integers in the set is composite since it is divisible by 3. The single exception to this rule is the case where the number divisible by 3 is the number 3 itself. Generally, we consider a set of n consecutive primes a trivial prime n-tuple if one of the primes of the set is *always* a factor of one of the terms of any sequence of consecutive odd integers having the same value for Δn.

3. The generalized problem. We are now in a position to state the generalized prime pair problem:

(1) For a given n, what is the value of Δn?
(2) For a given n, is the set of primes which yield Δn finite or infinite?

Clearly, the second question remains unanswered for all $n \geq 2$. The first question, however, can be answered for specific values of n by establishing the following conditions:

(A) Show that for *every* set of n odd primes $p_1, p_2, \cdots, p_n$ that $\Delta n > U$ where U is some positive integer.
(B) Exhibit a set of n odd primes such that $\Delta n = U + 2$.

4. An example. The procedure followed in establishing condition (A) will be illustrated for $n = 8$. Beginning with 1, write down 14 consecutive odd numbers (mod 10)

1 3 5 7 9 1 3 5 7 9 1 3 5 7.

Now, assuming the first number represents a prime, the 2nd or 3rd and every 3rd number thereafter is divisible by 3; one of the 2nd, 3rd, 4th, or 5th and every 5th number thereafter is divisible by 5, etc., for 7, 11, 13, $\cdots$. Taking every possible combination of assigning factors there will be one or more combinations which leave a maximum number of terms of the sequence which have not been assigned factors. Placing the factors which have been assigned underneath the terms to which they have been assigned, the maximum combination for this sequence is

1	3	5	7	9	1	3	5	7	9	1	3	5	7
		3			3		5	3			3	5	
		5			7							7	

We need not assign any further factors since 11, 13, 17, $\cdots$, can always be assigned in such a fashion as to involve only terms which have already been assigned factors. If the same procedure is followed for sequences of 14 odd numbers beginning with 3, 7, and 9, respectively, it is found that there is no combination that will leave *more* than 8 unassigned terms and two combinations that leave 8 unassigned terms, i.e.,

3	5	7	9	1	3	5	7	9	1	3	5	7	9
	5	3			3	5		3			3		
	7							7			5		

and

7	9	1	3	5	7	9	1	3	5	7	9	1	3
		3		5	3			3	5		3		

If we repeat the same process for 13 consecutive odd numbers we find there is no way to assign the factors to leave 8 terms unassigned. Since the difference between the first and 13th odd number is 24 we clearly have condition (A) satisfied when $U = 24$. Now, if we can find a set of 8 consecutive odd primes (mod 10) equal, respectively, to the 8 unassigned terms in any one of the three sequences

determined above we shall have proved that $\Delta 8 = 26$. The following primes complete the proof.

$$11, 13, 17, 19, 23, 29, 31, 37.$$

When conditions (A) and (B) have been established, the procedure described here gives *all* the essentially different poissible types of n-tuples for a given n and by solving one or more linear indeterminate equations the exact form of each type of n-tuple can be obtained. Thus, for $n = 8$, there are, at most, three different types of 8-tuples, viz.,

(a) $210x + 11, 13, 17, 19, 23, 29, 31, 37$
(b) $210x + 173, 179, 181, 187, 191, 193, 197, 199$
(c) $30x + 17, 19, 23, 29, 31, 37, 41, 43$.

5. Some further questions. The preceding discussion raises some new questions:

(3) Are there values of n such that $\Delta n > U + 2$?

The writer has verified that $\Delta n = U + 2$ for all $n \leq 10$. On the other hand an exhaustive search of the first two billion numbers failed to disclose a single 19-tuple of the types prescribed by the above procedure.

(4) Are there some possible types of n-tuples for which no n-tuple exists? For example, there are, at most, two types of 10-tuples; a 10-tuple for one type has been exhibited but not for the other; one possibility is that none exists for the other type.

6. Numerical results. The writer has computed Δn for $n \leq 26$ and determined all the possible types of n-tuples for each n in this range; these are tabulated in Table I. It is convenient to describe a given type of n-tuple by specifying the smallest prime in the sequence with respect to an appropriate modulus followed by the succession of differences between the constituent primes. It is observed that a kind of "duality" exists between the different types of n-tuples for a given n such that if the set of differences for a given n-tuple is known, its dual will have these same differences in reverse order. Some n-tuples are self dual and do not yield distinct types, such as $n = 6$. An examination of Table I shows that for $n = 2, 4$, and 6 there is only one type of n-tuple; for $n = 8$, there are 3 types; for $n = 3, 9, 15$, and 22 there are 4 types; for $n = 13$ there are 6 types; for all others in the table two types are given, each the dual of the other. In a recent attempt to discover another n-tuple that is self dual the writer ascertained that for $n = 41$ there are 8 distinct types of possible 41-tuples.

The numerical results obtained in this investigation were performed on an IBM 701 electronic calculator. Four separate programs were written. Program I was designed to search for 4-tuples; an exhaustive search of all numbers $\leq 5{,}073{,}379$ was made and 549 4-tuples were discovered.

Program II was designed to search for 7-tuples of the types

(A) $210x + 11, 13, 17, 19, 23, 29, 31$
(B) $210x + 179, 181, 187, 191, 193, 197, 199$.

Seventeen 7-tuples of type (A) and twenty-four of type (B) were discovered in an exhaustive search of all numbers $\leq 157{,}131{,}419$.

TABLE I

n	Δn	m	$p(\mathit{mod}\ m)$	*differences*
2	2	6	5	2
3	6	30	11	2,4
3	6	30	17	2,4
3	6	30	7	4,2
3	6	30	13	4,2
4	8	30	11	2,4,2
5	12	30	11	2,4,2,4
5	12	30	7	4,2,4,2
6	16	210	97	4,2,4,2,4
7	20	210	11	2,4,2,4,6,2
7	20	210	179	2,6,4,2,4,2
8	26	30	17	2,4,6,2,6,4,2
8	26	210	11	2,4,2,4,6,2,6
8	26	210	173	6,2,6,4,2,4,2
9	30	30	13	4,2,4,6,2,6,4,2
9	30	30	17	2,4,6,2,6,4,2,4
9	30	210	11	2,4,2,4,6,2,6,4
9	30	210	169	4,6,2,6,4,2,4,2
10	32	210	11	2,4,2,4,6,2,6,4,2
10	32	210	167	2,4,6,2,6,4,2,4,2
11	36	2310	1271	2,4,2,4,6,2,6,4,2,4
11	36	2310	1003	4,2,4,6,2,6,4,2,4,2
12	42	2310	1271	2,4,2,4,6,2,6,4,2,4,6
12	42	2310	997	6,4,2,4,6,2,6,4,2,4,2
13	48	2310	1271	2,4,2,4,6,2,6,4,2,4,6,6
13	48	2310	991	6,6,4,2,4,6,2,6,4,2,4,2
13	48	2310	1259	2,10,2,4,2,4,6,2,6,4,2,4
13	48	2310	1003	4,2,4,6,2,6,4,2,4,2,10,2
13	48	2730	1199	2,6,6,4,2,4,6,2,6,4,2,4
13	48	2730	1483	4,2,4,6,2,6,4,2,4,6,6,2
14	50	30030	15131	2,4,2,4,6,2,6,4,2,4,6,6.2
14	50	30030	14849	2,6,6,4,2,4,6,2,6,4,2,4,2
15	56	30030	15131	2,4,2,4,6,2,6,4,2,4,6,6,2,6
15	56	30030	14843	6,2,6,6,4,2,4,6,2,6,4,2,4,2
15	56	210	17	2,4,6,2,6,4,2,4,6,6,2,6,4,2
15	56	210	137	2,4,6,2,6,6,4,2,4,6,2,6,4,2
16	60	30030	6943	4,2,4,6,2,6,4,2,4,6,6,2,6,4,2
16	60	30030	23027	2,4,6,2,6,6,4,2,4,6,2,6,4,2,4
17	66	30030	6943	4,2,4,6,2,6,4,2,4,6,6,2,6,4,2,6
17	66	30030	23021	6,2,4,6,2,6,6,4,2,4,6,2,6,4,2,4
18	70	30030	6943	4,2,4,6,2,6,4,2,4,6,6,2,6,4,2,6,4
18	70	30030	23017	4,6,2,4,6,2,6,6,4,2,4,6,2,6,4,2,4
19	76	5 10510	2 17153	4,2,4,6,2,6,4,2,4,6,6,2,6,4,2,6,4,6
19	76	5 10510	2 93281	6,4,6,2,4,6,2,6,6,4,2,4,6,2,6,4,2,4
20	80	30030	29	2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,4,2,4,2
20	80	30030	29921	2,4,2,4,8,6,4,6,2,4,6,2,6,6,4,2,4,6,2
21	84	30030	29	2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,4,2,4,2,4
21	84	30030	29917	4,2,4,2,4,8,6,4,6,2,4,6,2,6,6,4,2,4,6,2
22	90	5 10510	19	4,6,2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,4,2,4,2
22	90	5 10510	5 10401	2,4,2,4,8,6,4,6,2,4,6,2,6,6,4,2,4,6,2,6,4
22	90	5 10510	23	6,2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,4,2,4,2,4
22	90	5 10510	5 10397	4,2,4,2,4,8,6,4,6,2,4,6,2,6,6,4,2,4,6,2,6
23	94	5 10510	19	4,6,2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,4,2,4,2,4
23	94	5 10510	5 10397	4,2,4,2,4,8,6,4,6,2,4,6,2,6,6,4,2,4,6,2,6,4
24	100	5 10510	2 17153	4,2,4,6,2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,4,6,2,4
24	100	5 10510	2 93257	4,2,6,4,8,6,4,6,2,4,6,2,6,6,4,2,4,6,2,6,4,2,4
25	110	30030	29	2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,4,2,4,2,4,14,4,6,2
25	110	30030	29891	2,6,4,14,4,2,4,2,4,8,6,4,6,2,4,6,2,6,6,4,2,4,6,2
26	114	96 99690	34 64999	2,12,4,2,4,6,2,6,4,2,4,6,8,6,4,2,6,4,6,8,4,2,4,2,4
26	114	96 99690	62 34577	4,2,4,2,4,8,6,4,6,2,4,6,8,6,4,2,4,6,2,6,4,2,4,12,2

TABLE II

1st term of n-tuple	n	Type
11	10	A
1277	9	C
5639	7	B
88789	9	B
1 13143	9	C
1 13147	9	C
1 65701	7	A
2 84723	8	B
6 26609	7	B
8 55709	9	B
10 68701	7	A
11 46773	8	B
25 80647	8	C
65 60993	8	B
75 40439	7	B
85 73429	7	B
119 00501	7	A
157 60091	8	A
178 43459	7	B
185 04371	7	A
190 89599	7	B
207 37877	8	C
210 36131	7	A
240 01709	7	B
256 58441	8	A
394 31921	7	A
429 81929	7	B
435 34019	7	B
450 02591	7	A
678 16361	7	A
691 56533	8	B
733 73537	8	C
742 66249	9	B
761 70527	8	C
792 08399	7	B
804 27029	7	B
841 04549	7	B
868 18211	7	A
879 88709	7	B
936 25991	8	A
1006 58627	8	C
1240 66079	7	B
1247 16071	7	A
1284 69149	7	B
1347 64997	8	C
1362 61241	7	A
1379 43347	8	C
1401 17051	7	A
1442 14319	7	B
1546 35191	7	A
1571 31419	7	B

Program III was designed to search for 8-tuples of the type

(C) $30x + 17, 19, 23, 29, 31, 37, 41, 43.$

Ten 8-tuples of this type were obtained in a search of all numbers $\leq 137,943,347$.

Program IV was the unsuccessful attempt to find a 19-tuple. As a check on the above calculations program II was recently re-run on an IBM 650 magnetic drum calculator; there was 100 percent agreement between the results of the two machines.

An examination of Table I shows that all n-tuples in a given range of numbers for $7 \leq n \leq 15$ can be determined by examining only the n-tuples obtained from the three types given above. This subsequent calculation yielded three 8-tuples and one 10-tuple for type (A), four 8-tuples and three 9-tuples for type (B) and three 9-tuples for type (C). These results are tabulated in Table II. Since a 9-tuple may also be an 8-tuple, and a 7-tuple, etc., the highest ranking designation is given in each case.

7. Conclusion. The prime pair problem has been shown to be the simplest case of a more general problem. Essentially, the problem poses two questions:

(5) How "clustered" can a given number of primes be?

(6) How many such "clusters" are there?

The first question is considered equivalent to the question, what is the smallest range of natural numbers that can contain n primes? The scheme proposed in this paper gives a necessary range; it is suggested but not proved generally that this range is sufficient. It is proved to be both necessary and sufficient for all n-tuples for $n \leq 10$.

G. H. Hardy [3] and W. W. Ball [4] have made conjectures concerning the number of prime pairs and higher n-tuples under a given magnitude but the results of this investigation are too meager to expect any degree of accuracy from the formulas that have been proposed.

The programs for this investigation were not the most efficient since the writer had a limited amount of time to write, debug and run them. With a more elegant approach to the programming and by using the more sophisticated logic of the IBM 704 these calculations could be accelerated by a factor of 200 or more.

The writer wishes to express his thanks to the referee for his comments and suggestions.

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Generation of Bessel Functions on High Speed Computers

The generation of values of sets of functions such as the Bessel functions $J_n(x)$ and $Y_n(x)$ is a problem which one encounters frequently in numerical calculations. The present paper is concerned with describing a method which has been found particularly useful on high speed computers. The basic idea is due to J. C. P. Miller and is described in detail [as applied to the Bessel functions $I_n(x)$] in the introduction to the second volume [1] of Bessel functions published by BAAS. The tables in the BA volume were computed by hand as were the tables by Fox [2]. However, the method is particularly suited to large scale computers. In preparing a general purpose routine to compute $J_n(x)$ and $Y_n(x)$ one is faced with various obstacles. The power series representation of $J_n(x)$ is useful for values of x which are not too large. In addition to the slowness of convergence the form of $Y_n(x)$ becomes complicated. The asymptotic expansions of $J_n(x)$ and $Y_n(x)$ are useful when x is large with respect to n. In the region where both n and x are large neither of these representations is useful and a third form must be used. It is clear that any program which employs these representations and which should be generally useful must automatically choose the proper form for computation. The method of Miller avoids these difficulties and provides a simple algorithm not only for $J_n(x)$ but for many other functions with a similar behavior.

Specifically, the method is as follows (the theory supporting the method is given in [1], p. xvii):

1. Choose k larger than the greater of n or x, assume $\bar{J}_{k+1} = 0$, $\bar{J}_k = \alpha \neq 0$, where α is some arbitrarily chosen constant.

2. If $z = 1/x$, generate the sequence $\bar{J}_{k-1}, \bar{J}_{k-2}, \cdots, \bar{J}_1, \bar{J}_0$ from the recurrence relation

$$\bar{J}_{p-1} = 2pz\bar{J}_p - \bar{J}_{p+1}$$

starting with $p = k$. If k is chosen sufficiently large we will obtain $J_p = c\bar{J}_p$ for desired values of p ranging from $p = 0$ to $p = n$.

3. To determine the constant c use the functional relation

$$J_0 + 2\sum_{m=1}^{\infty} J_{2m} = 1.$$

The choice of k offers no difficulty since the calculation may be done iteratively. Thus, starting with a value of $k > \max(n, x)$ we perform the calculations in (2) and (3) once and then repeat with k increased by a fixed amount. The results obtained are compared in accordance with a preassigned tolerance and the process is repeated until the criteria for acceptance are satisfied. In practice, if the values of J_p are desired for $p = 0, 1, \cdots, n$ the comparison need only be made at $p = n$. It is advisable to employ floating point arithmetic and choose α as small as possible.

4. To generate the functions $Y_n(x)$ we employ the following representation for $Y_0(x)$,

$$Y_0(x) = (2/\pi)\left[J_0(x)\left\{\log\frac{x}{2} + \gamma\right\} + 2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}J_{2m}(x)}{m}\right]$$

where γ is Euler's constant. We note that the series part here employs the same values J_{2k} as occur in the calculation of the normalization constant c.

5. Compute $Y_1(x)$ from the relation

$$J_1(x)Y_0(x) - J_0(x)Y_1(x) = 2/\pi x.$$

6. Generate $Y_p(x)$ in a forward direction for $p = 1, 2, \cdots, n - 1$ from the recurrence relation

$$Y_{p+1} = 2pzY_p - Y_{p-1}.$$

The method may be employed in a similar manner for other sets of functions. In the following we outline the procedure for some of the functions for which it may be employed.

Modified Bessel Functions $I_n(x)$ *and* $K_n(x)$. Here the recurrence relation (descending order) for $I_n(x)$ is

$$I_{p-1} = 2pzI_p + I_{p+1}$$

while the normalization can be obtained from the relation

$$I_0 + 2\sum_{m=1}^{\infty} I_m = e^x.$$

While there is an analogous relation for $K_0(x)$ corresponding to that for Y_0, the possible loss of significant digits makes it unfeasible to use. To compute $K_0(x)$ the integral representation

$$K_0(x) = \int_0^\infty \exp(-x \cosh u)\, du$$

will be convenient for all values of $x > .01$. (Alternatively one may use polynomial approximations given by Allen [3].) The relation

$$K_1I_0 + K_0I_1 = 1/x$$

will yield K_1 while the recurrence relation (ascending order)

$$K_{p+1} = 2pzK_p + K_{p-1}$$

may be used to generate as many values of K_{p+1} as may be desired.

Spherical Bessel Functions $j_n(x)$ *and* $n_n(x)$. The spherical Bessel Functions $j_n(x) = \sqrt{\pi/2x}\, J_{n+\frac{1}{2}}(x)$ may be generated with the aid of the recurrence relation (descending order)

$$j_{p-1} = (2p + 1)zj_p - j_{p+1}$$

with the normalization constant determined from the fact that $j_0 = \sin x/x$. To determine the function $n_n(x) = (-1)^{n+1}\sqrt{\pi/2x}\, J_{-(n+\frac{1}{2})}(x)$ we have

$$n_0 = -\frac{\cos x}{x} \text{ and } n_1 = -\frac{\sin x}{x} - \frac{\cos x}{x^2}$$

and it is convenient to compute these functions directly and compute as many further values of n_p as desired with the previous recurrence relation (ascending order) which is also satisfied by n_p.

Modified Bessel Functions of Half-Integral Order. The functions

$$i_n = \sqrt{\pi/2x}\, I_{n+\frac{1}{2}}(x)$$

$$k_n = (-1)^{n+1} \sqrt{\pi/2x}\, K_{n+\frac{1}{2}}(x)$$

both satisfy the recurrence relation

$$(2p + 1)zi_p = i_{p-1} - i_{p+1}$$

and may be computed in a fashion entirely similar to the computation of the spherical Bessel functions. We have also

$$i_0 = \frac{\sinh x}{x}$$

$$k_0 = -\frac{\pi}{2x} e^{-x}$$

$$k_1 = \frac{\pi e^{-x}(x + 1)}{2x^2}.$$

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Remarks on the Disposition of Points in Numerical Integration Formulas

1. Introduction. Numerical integration formulas of degree 2 (i.e., exact for polynomials of at most degree 2) consisting of $n + 1$ equally weighted points have been developed for certain regions in n-dimensional Euclidean space. Thacher [1] discusses the equations which a formula of this type must satisfy for regions which are invariant under the group of linear transformations which leave the n-cube with vertices $(\pm a, \pm a, \cdots, \pm a)$ invariant; we call these symmetric regions. Hammer and Stroud [2] give two such formulas for the n-simplex. Hammer [3] has shown that a set of $2n$ equally weighted points lying on the coordinate axes form a formula of degree 3 for any symmetric region.

In section 2 of this note we show that the formulas of degree 2 discussed by Thacher can be described geometrically. We also show that there is a similar class of formulas for the regular n-simplex. In section 3 we describe geometrically a wide class of formulas of degree 3 containing $2n$ points for symmetrical regions.

2. Formulas of degree 2. This section is devoted to the proof of the following:

THEOREM 1. *A necessary and sufficient condition that $n+1$ equally weighted points form a numerical integration formula of degree 2 for a symmetric region or for a regular n-simplex is that these points form the vertices of a regular n-simplex whose centroid coincides with the centroid of the region and lie on the surface of a sphere of radius* $r = \sqrt{nI_2/I_0}$, *where*

$$I_0 = \int_R dv \qquad I_2 = \int_R x_1^2 dv = \cdots = \int_R x_n^2 dv.$$

Proof. The weight for the points is $I_0/(n+1)$. We first prove the theorem for symmetrical regions; the proof for the n-simplex follows easily. Let R be a symmetrical region. We then have

$$\int_R x_i dv = \int_R x_i x_j dv = 0 \qquad i, j = 1, \cdots, n \qquad i \neq j.$$

Now suppose the $n+1$ points

$$v_i = (v_{i1}, v_{i2}, \cdots, v_{in}) \qquad i = 0, 1, \cdots, n \tag{1}$$

are an integration formula of degree 2 for R with equal weights. Then they satisfy the equations

$$v_{0i} + v_{1i} + \cdots + v_{ni} = 0 \qquad i = 1, \cdots, n \tag{2}$$

$$v_{0i}v_{0j} + v_{1i}v_{1j} + \cdots + v_{ni}v_{nj} = \frac{(n+1)I_2}{I_0}\delta_{ij} \qquad i, j = 1, \cdots, n. \tag{3}$$

To show that the points v_i satisfy the conditions of the theorem consider the matrix

$$A = \begin{bmatrix} v_{01} & v_{11} & v_{21} & \cdots & v_{n1} \\ v_{02} & v_{12} & v_{22} & \cdots & v_{n2} \\ \cdot & \cdot & \cdot & & \cdot \\ v_{0n} & v_{1n} & v_{2n} & \cdots & v_{nn} \\ \sqrt{I_2/I_0} & \sqrt{I_2/I_0} & \sqrt{I_2/I_0} & \cdots & \sqrt{I_2/I_0} \end{bmatrix}$$

By (2) and (3) it is seen that

$$AA^T = \frac{(n+1)I_2}{I_0} I$$

where I is the identity matrix. Hence

$$A^T A = \frac{(n+1)I_2}{I_0} I$$

is equivalent to the equations

$$v_{i1}v_{j1} + \cdots + v_{in}v_{jn} + \frac{I_2}{I_0} = \frac{(n+1)I_2}{I_0}\delta_{ij} \qquad i, j = 0, 1, \cdots, n. \tag{4}$$

This shows that the points lie on the sphere of radius $\sqrt{nI_2/I_0}$ with centroid at the origin. To show that the v_i are vertices of a regular n-simplex it suffices to

show that they are equidistant. By (4)

$$d^2(\nu_i, \nu_j) = \nu_{i1}^2 + \cdots + \nu_{in}^2 + \nu_{j1}^2 + \cdots + \nu_{jn}^2 - 2(\nu_{i1}\nu_{j1} + \cdots + \nu_{in}\nu_{jn})$$

$$= \frac{2(n+1)I_2}{I_0}.$$

Reversal of the above argument proves the condition of the theorem is sufficient. This establishes the theorem for symmetric regions.

To show the theorem holds for the n-simplex consider a particular set of $n+1$ equidistant points which lie on an n-sphere with center at the origin and radius $r = a\sqrt{1/(n+2)}$. The results in [2] show that there exists a regular n-simplex S_n for which these points are an integration formula. The vertices of S_n lie on a sphere of radius a. The above argument shows that any $n+1$ equidistant points on the sphere of radius r is also an integration formula for S_n. This completes the proof of the theorem.

TABLE 1

Region	I_2/I_0
P_n	$a^2/(n+2)$
S_n	$a^2/n(n+2)$
C_n	$a^2/3$
Q_n	$2a^2/(n+1)(n+2)$

Values of I_2/I_0 are given in Table 1 for four regions (i.e., their interiors). P_n is the n-sphere of radius a with centroid at the origin. S_n is any regular n-simplex whose vertices lie on P_n. C_n is the n-cube with vertices $(\pm a, \pm a, \cdots, \pm a)$ or this cube rotated in any manner. Q_n is the region defined by the 2^n inequalities $\pm x_1 \pm x_2 \pm \cdots \pm x_n \leq a$, or any rotation of this region. The regions S_n, C_n, and Q_n are the only regular polytopes for $n \geq 5$ ([4], p. 120). In the following section we use the fact that any vertex of Q_n is a distance of $a\sqrt{2}$ from $2(n-1)$ vertices and a distance of $2a$ from one vertex.

3. Formulas of degree 3. In this section we prove

THEOREM 2. *A necessary and sufficient condition that $2n$ points $\nu_1, \cdots, \nu_n, -\nu_1, \cdots, -\nu_n$ form an equally weighted numerical integration formula of degree 3 for a symmetrical region is that these points form the vertices of a Q_n whose centroid coincides with the centroid of the region and lie on an n-sphere of radius $r = \sqrt{nI_2/I_0}$.*

Proof. The weight is $\frac{1}{2n} I_0$. Let the points

$$(5) \qquad \nu_i = (\nu_{i1}, \cdots, \nu_{in}) \qquad -\nu_i = (-\nu_{i1}, \cdots, -\nu_{in}) \qquad i = 1, \cdots, n$$

satisfy

$$(6) \qquad \nu_{1i}\nu_{1j} + \cdots + \nu_{ni}\nu_{nj} = \frac{nI_2}{I_0}\delta_{ij} \qquad i, j = 1, \cdots, n.$$

(The integrals of the odd degree monomials are zero and are identically satisfied.) Setting

$$A = \begin{bmatrix} \nu_{11} & \nu_{21} & \cdots & \nu_{n1} \\ \nu_{12} & \nu_{22} & \cdots & \nu_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ \nu_{1n} & \nu_{2n} & \cdots & \nu_{nn} \end{bmatrix}$$

the proof follows as in theorem 1. We find

$$d^2(\pm\nu_i, \pm\nu_j) = \frac{2nI_2}{I_0} \qquad i \neq j \qquad i, j = 1, \cdots, n$$

and

$$d^2(\nu_i, -\nu_i) = \frac{4nI_2}{I_0}.$$

This completes the proof.

The results of theorems 1 and 2 generalize results given by Thacher for the 2-cube (square) and the 3- cube. There may exist unequally weighted integration formulas containing fewer points than those discussed here; however, the equally weighted formulas will probably remain important for applications.

4. Concluding remarks. One of the properties desirable in an integration formula is that the points be interior to the region. The $2n$ point formula given by Tyler [5] for C_n using points on the coordinate axes has the points outside for $n > 3$. We give here two formulas for C_n (with vertices $(\pm 1, \pm 1, \cdots, \pm 1)$) of degrees 2 and 3 for which the points are interior for all n ([4], 245):

Let Γ_k denote the point $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ where

$$\gamma_{2r-1} = \sqrt{\frac{2}{3}} \cos \frac{2rk\pi}{n+1} \qquad \gamma_{2r} = \sqrt{\frac{2}{3}} \sin \frac{2rk\pi}{n+1}$$

$$r = 1, 2, \cdots, [\tfrac{1}{2}n]$$

($[\frac{1}{2}n]$ is the greatest integer not exceeding $\frac{1}{2}n$), and if n is odd $\gamma_n = (-1)^k/\sqrt{3}$. Then $\Gamma_0, \Gamma_1, \cdots, \Gamma_n$ satisfy the conditions of theorem 1, and all are interior to C_n.

Let Σ_k denote the point $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ where

$$\sigma_{2r-1} = \sqrt{\frac{2}{3}} \cos \frac{(2r-1)k\pi}{n} \qquad \sigma_{2r} = \sqrt{\frac{2}{3}} \sin \frac{(2r-1)k\pi}{n}$$

$$r = 1, 2, \cdots, [\tfrac{1}{2}n]$$

and if n is odd $\sigma_n = (-1)^k/\sqrt{3}$. Then $\Sigma_1, \cdots, \Sigma_{2n}$ satisfy the conditions of theorem 2 and all are interior to C_n.

For the formulas and regions we have considered, only the location of the points relative to each other and not to the region was found to be important. The insight given by these results may prove useful in further investigations; one problem is that concerning formulas for more general regions.

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TECHNICAL NOTES AND SHORT PAPERS

A Method for the Numerical Evaluation of Certain Infinite Integrals

The solution of many physical problems often necessitates the numerical evaluation of infinite real integrals, a common example being that of solutions obtained with the aid of integral transforms. The evaluation of such integrals is often a laborious task, particularly if the integrand is oscillatory, so that it is usual to resort to special methods which give information for certain ranges of values of the variables; methods of this type are those involving asymptotic expansions or the related techniques of steepest descent and of stationary phase. The purpose of the present note is to outline a method in which the value of such integrals is expressed in terms of a convergent series obtained by a modification of the corresponding asymptotic expansion. The development is given below for a special case only, namely one which might arise in conjunction with the use of sine transforms; it will be clear however that these results can be readily generalized to other types of integrals which are usually reduced to an asymptotic representation. Examples may be found in Erdélyi [1]. The method is thus valid whether the integrand is oscillatory or not; in fact, though the special integrand considered in detail below does oscillate, inspection of the convergence proofs shows that this fact is of little importance to the developments presented. A method which holds in the case of oscillatory integrands has been described by I. M. Longman [2].

Basic expansions. Consider a convergent integral $I(a)$ of the form

$$(1) \qquad I(a) = \int_a^\infty f(x) \sin x dx; \qquad f(x) \to 0 \text{ steadily as } x \to \infty.$$

By $f(x) \to 0$ steadily, we mean that $f(x_1) \geq f(x_2) > 0$ if $x_1 < x_2$ and $\lim f(x) = 0$; see Whittaker and Watson [3]. N successive integrations by parts may be shown to give the following result

$$(2) \qquad I(a) = \sum_{i=0}^{N} f^{(i)}(a) \cos [a + i(\pi/2)] + \int_a^\infty f^{(N)}(x) \sin [x + N(\pi/2)]dx$$

where $f^{(i)} = (d^i f/dx^i)$, provided that $f(x)$ is differentiable the required number of times, and that

$$(2a) \qquad f^{(i)}(x) \to 0 \text{ steadily as } x \to \infty; \qquad i = 0, 1, 2, \cdots.$$

The term in equation (2) containing the summation usually represents an asymptotic representation of I for large values of a, and the infinite series obtained as N is increased indefinitely in general does not converge. A convergent expansion

for $I(a)$ may now be derived in the following manner. Integration by parts gives

$$\text{(3a)} \qquad I(a) = \int_a^{a_1} f(x) \sin x dx + f(a_1) \cos a_1 + \int_{a_1}^{\infty} f^{(1)}(x) \cos x dx$$

and further

$$\text{(3b)} \quad I(a) = \int_a^{a_1} f(x) \sin x dx + \int_{a_1}^{a_2} f^{(1)}(x) \cos x dx + f(a_1) \cos a_1 - f^{(1)}(a_2) \sin a_2 - \int_{a_2}^{\infty} f^{(2)}(x) \sin x dx.$$

Repetition of this process finally gives

$$\text{(4)} \quad I(a) = \sum_{i=0}^{\infty} f^{(i)}(a_{i+1}) \cos [a_{i+1} + i(\pi/2)] + \sum_{i=0}^{\infty} \int_{a_i}^{a_{i+1}} f^{(i)}(x) \sin [x + i(\pi/2)] dx$$

where one may set

$$\text{(4a)} \qquad a_{i+1} \geq a_i; \qquad a_0 = a.$$

It will now be shown that the quantities a_i may be chosen in such a manner that the two series on the right-hand side of equation (4) converge.

Convergence of series expansion. The first series on the right-hand side of equation (4) will certainly converge if the a_i's are chosen so that the series

$$\text{(5)} \qquad S_1 = \sum_{i=0}^{\infty} f^{(i)}(a_{i+1})$$

converges; and this series will converge (absolutely) if a positive number ρ independent of i exists such that

$$\text{(5a)} \qquad 1 > \rho > | f^{(i)}(a_{i+1})/f^{(i-1)}(a_i) |$$

for all $i \geq 1$. It will now be shown that such a choice of a_i's is always possible. (The author is indebted to Dr. C. C. Chao for his valuable suggestions concerning this proof.)

Choose the quantity $a_i \geq a_0$ arbitrarily; then the value of $f^{(0)}(a_1)$ is known and a_2 must be selected so that

$$\text{(5b)} \qquad | f^{(1)}(a_2) | < \rho | f^{(0)}(a_1) |$$

as may always be done because of relation (2a). Now however the value of $f^{(1)}(a_2)$ is known, and so a_3 can be chosen by a similar procedure. Repetition of this process yields values of all a_i's in such a manner that relation (5a) is satisfied for all $i \geq 1$ and therefore series S_1 converges absolutely. It should be noted that the choice of a_i's is not unique, and that in fact if such a choice has been made ($a_i = a_i'$, say) then the values $a_i = a_i''$ will also insure convergence of S_1 provided only that

$$\text{(6)} \qquad a_i'' \geq a_i'$$

in view of the steadiness requirement of equation (2a).

It will now be shown that the a_i's may be taken in conformity with requirement (6) and, in addition, so that the second series of equation (4), namely

$$(7) \qquad S_2 = \sum_{i=0}^{\infty} I_i; \qquad I_i = \int_{a_i}^{a_{i+1}} f^{(i)}(x) \sin[x + i(\pi/2)]dx$$

also converges. Note first that it follows from equation (2a) that, for any i, a number A_i exists such that

$$(7a) \qquad |f^{(i-1)}(x_i)| < |f^{(i-1)}(x)| \qquad \text{for all } x_1 > x > A_i.$$

Let now the quantities a_i be selected (consistently with inequality (6)), so that

$$(7b) \qquad a_i > A_i.$$

Because of equation (4a) then the relation

$$(7c) \qquad |f^{(i-1)}(a_{i+1})/f^{(i-1)}(a_i)| < 1$$

holds for all i.

Consider now the integrals I_i; because of the steadiness requirement in equation (2a) the quantity $f^{(i)}(x)$ does not change sign within $a_i \le x \le a_{i+1}$ and

$$(8a) \qquad |I_i| < \left| \int_{a_i}^{a_{i+1}} f^{(i)}(x)dx \right| = |f^{(i-1)}(a_{i+1}) - f^{(i-1)}(a_i)| =$$

$$|f^{(i-1)}(a_i)|\,1 - [f^{(i-1)}(a_i+1)/f^{(i-1)}(a_i)]\,| < 2\,|f^{(i-1)}(a_i)|\,; \qquad i \ne 0$$

in view of relation (7c). Series S_2 (with the possible omission of the first term) is then term-by-term less than the series

$$(8b) \qquad 2\sum_{i=1}^{\infty} |f^{(i-1)}(a_i)| = 2\sum_{i=0}^{\infty} |f^{(i)}(a_{i+1})|$$

which has been shown to converge. Hence S_2 also converges.

Example. As an illustration of the procedure indicated above, the special case of $f(x) = x^{-k}$ will be considered; thus

$$(9) \qquad I(a) = \int_a^{\infty} x^{-k} \sin x dx; \qquad k > 0.$$

Here one may take (as will be shown)

$$(10) \qquad a_i = a + i\alpha$$

where α is a constant; equation (4) then reduces to

$$(10a) \qquad I(a) = S_1(a) + S_2(a)$$

where

$$S_1(a) = \sum_{i=1}^{\infty} \frac{(1)(k)(k+1)\cdots(k+i-2)}{(a+i\alpha)^{(k+i-1)}} \sin[i(\pi/2 - \alpha) - a]$$

(10b)

$$S_2(a) = \sum_{i=1}^{\infty} (1)(k)(k+1)\cdots(k+i-2)\int_{a+(i-1)a}^{a+i\alpha} x^{-(k+i-1)} \cos[i(\pi/2) - x]dx.$$

Series S_1 converges if

$$\text{(10c)}\quad 1 > \lim_{i\to\infty}\left\{\frac{(k+i-1)(a+i\alpha)^{(k+i-1)}}{[a+(i+1)\alpha]^{(k+i)}}\right\}$$

$$= \lim_{i\to\infty}\left(\frac{k+i-1}{a+i\alpha}\right)\lim_{i\to\infty}\left(\frac{a+i\alpha}{a+(i+1)\alpha}\right)^k\lim_{i\to\infty}\left(\frac{a+i\alpha}{a+(i+1)\alpha}\right)^i$$

$$= (1/\alpha)\lim_{i\to\infty}\left(1-\frac{1}{1+(a/\alpha)+i}\right)^i = 1/(\alpha e)$$

or in other words if

$$\text{(10d)}\qquad \alpha > (1/e).$$

Series S_2 may now be considered by expanding the integrals it contains in a manner entirely analogous to that of equations (8a) and (8b), and it can thus be easily shown that this series also converges if α is chosen as specified in equation (10d); the latter condition then represents in the present case the only requirement for convergence of expansion (10a).

An advantageous choice of α, consistent with requirement (10d), is $\alpha = \pi/2$, since in this case the sine-term in S_1 is constant; in particular, if $a = 0$ (or $a = n\pi$), note that $S_1 = 0$. No choice of α is of course possible which will make $S_2 = 0$, so that numerical evaluations of integrals are still necessary. Series S_2 converges quite rapidly, however; as an example consider in fact the case of $a = 0$, $k = 1$ and $\alpha = \pi/2$ for which the value of the integral in question is well known. The result may be written as

$$\text{(11)}\quad (2/\pi)\int_0^\infty (1/x)\sin x\,dx = 1 = (2/\pi)\sum_{i=1}^{\infty}(i-1)!\int_{(i-1)\pi/2}^{i\pi/2} x^{-i}\cos[i(\pi/2) - x]dx.$$

The integrals in this summation were evaluated by Simpson's rule with the relatively coarse interval of $(\pi/8)$. The value of the summation itself may be expressed as the limit of the sequence of the partial sums S_i of the first i terms of the series; the first few terms of this sequence were found to be as follows (to four significant figures):

$$\text{(11a)}\quad S_1 = .8727;\quad S_2 = .9762;\quad S_3 = .9951;\quad S_4 = .9988;\quad S_5 = .9996$$

and may therefore be said to converge fairly rapidly. Almost the same results were obtained when the coarser interval of $(\pi/4)$ was used in Simpson's rule.

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A Formula for the Approximation of Definite Integrals of the Normal Distribution Function

A simple expression has been found which may be used to yield approximate numerical values for definite integrals of the normal distribution function. The expression is as follows.

$$\int_X^\infty e^{-t^2/2}dt = \frac{e^{-X^2/2}}{X + .8e^{-.4X}}\ ; X \geq 0.$$

This approximation is satisfactory for certain applications over a broad range of values for the finite integration limit, as may be judged from the table below.

X	$\int_X^\infty \exp(-t^2/2)dt$	$\dfrac{\exp(-X^2/2)}{X + .8\exp(-.4X)}$	Difference	% Difference
0	1.253	1.250	−.003	− .2
.2	1.055	1.044	−.011	−1.0
.5	.773	.764	−.009	−1.2
1.0	.398	.395	−.003	− .8
1.5	.1675	.1674	−.0001	0
2.0	.0570	.0573	+.0003	+ .5
2.5	.0156	.0157	.0001	.6
3.0	.00338	.00343	.00005	1.5
3.5	.00058 3	.00059 2	.00000 9	1.5
4.0	.00007 95	.00008 06	.00000 11	1.4
5.0	.00000 0718	.00000 0730	.00000 0012	1.7

Some of the properties of Mill's ratio, the function here approximated by $X + .8\,e^{-.4X}$, have been recently described by Sampford [1].

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1. M. R. SAMPFORD, "Some inequalities on Mill's ratio and related functions," *Annals Math. Stat.*, 1953, v. 24, p. 130.

Some Factorizations of Numbers of the Form $2^n \pm 1$

The author has prepared a factorization routine for use on an IBM 701 computer. In this note, we describe the routine briefly, and report on some results obtained during the period February–April 1957 on the computer at the University of California, Berkeley.

In the basic routine, an arithmetic progression is given in which divisors of a number N are to be sought. Only single word divisors, that is, divisors less than 2^{35}, are considered, but the number N may be many words. After deleting those terms of the progression which are multiples of 2, 3, 5, 7, or 11, the remaining terms up to $N^{\frac{1}{2}}$, or up to a prescribed bound, are tried as divisors of N. In order to delete the multiples of 2, 3, 5, 7, and 11 efficiently, use is made of the fact that the differences of the remaining terms of the progression repeat with a period $\phi(2\cdot3\cdot5\cdot7\cdot11) = 480$. The 480 required differences are computed in advance, and used repeatedly.

There are variations of the routine which make it possible to use the fact that ± 2 is a quadratic residue mod N, and which expedite handling very large numbers, several related numbers, or sequences of numbers. For example, using the last of these, it was easy to find that the primes nearest to 2^{35} on either side are $2^{35} - 31$ and $2^{35} + 53$.

The routine can theoretically factor any number $N < 2^{70}$. But if nothing is known about the form of the factors, the maximum possible time (which increases roughly with $N^{\frac{1}{2}}$) would be prohibitive for the larger values of N. However, any number $N < 2^{36}$ can be factored in less than a minute, and any number $N < 2^{48}$ can be factored in less than an hour.

The remainders from the various divisions are summed and the sum is recorded at intervals. If a re-run is made and the remainder sums agree, this indicates that no machine errors have occurred.

The first work undertaken was a search for factors of the early Fermat numbers, supplementing work of a somewhat different character done previously [4], and extending the results of Selfridge [5]. It is known that any prime factor p of the Fermat number $F_m = 2^{2^m} + 1$ must satisfy the congruence $p \equiv 1 \pmod{2^{m+2}}$ if $m > 1$. All possible divisors $p < 2^{32}$ of any Fermat number were tried, and also all possible divisors with $p < 2^{35}$ and $p \equiv 1 \pmod{2^{15}}$. The total running time was about five hours. No new divisors were found. The computation was checked by a re-run. Thus we can say definitely that there are no factors in this range beyond those previously found, all of which have been known for more than thirty years, except the two found by Selfridge [5]. In particular, F_7 has no factor $p < 2^{32}$; since it is composite, it must be the product of either 2 or 3 prime factors. Also, we can say that F_{13}, whose character is unknown, has no factor less than 2^{35}. (After the computation described above was finished, I learned that some unpublished work of Selfridge, extending [5], had covered most of the same range. However, the work had not all been checked, and there were some gaps. Selfridge also tried some divisors between 2^{35} and 2^{36}; I was unable to check this part of his work.)

Another project was the study of the Mersenne numbers $2^n - 1$, where n itself is a Mersenne prime, that is, $n = 2^m - 1$, where m and n are prime. According to a well-known conjecture, any such number is prime. An extremely long computation of D. J. Wheeler (see [3], page 844) indicated that $2^{8191} - 1$ (corresponding to $m = 13$) is composite, and the conjecture therefore false. However, no factor of this number was found. It was decided to make a search for factors $p < 2^{35}$ of such numbers. The factors must satisfy $p \equiv 1 \pmod{2n}$ and $p \equiv 1, 7 \pmod 8$. The only values of m in question are $m = 13, 17, 19, 31$. Two factors were found, namely

$$1768(2^{17} - 1) + 1 \mid 2^{2^{17}-1} - 1$$

and

$$120(2^{19} - 1) + 1 \mid 2^{2^{19}-1} - 1.$$

These results can easily be verified using a desk calculator. Thus the conjecture is certainly false. The time required for the test was about an hour and a half. The computation was checked by a re-run. Thus we are sure that there are no other factors in the range considered. In particular, $2^{8191} - 1$ has no factor less than 2^{35}.

Finally, factorization of various numbers of the forms $2^n - 1$ and $2^n + 1$, which had not previously been factored, was attempted. (A table of the factors, known in 1925, of numbers of these forms with $n < 500$ appears in Cunningham and Woodall [1], and a number of additions to this table were given by Lehmer [2]. A few other factorizations have appeared elsewhere.) A somewhat arbitrary selection of cases was made, consisting of the following:

(a) $2^n - 1$ with n odd. The primitive factors satisfy $p \equiv 1 \pmod{2n}$ and $p \equiv 1, 7 \pmod 8$. The cases tried were $n = 95, 97, 101, 103, 109, 119, 121, 125, 129, 131, 133, 137, 139, 149, 157$.

(b) $2^n + 1$ with n odd. The primitive factors satisfy $p \equiv 1 \pmod{2n}$ and $p \equiv 1, 3 \pmod 8$. The cases tried were $n = 71, 101, 103, 107, 109, 113, 115$.

(c) $2^n + 1$ with $n \equiv 0 \pmod 4$. The primitive factors satisfy $p \equiv 1 \pmod{4n}$. The cases tried were $n = 104, 112, 116, 124$. (For the cases $n = 128, 256$, etc., see the discussion of Fermat numbers above.)

(d) $2^n + 1$ with $n \equiv 2 \pmod 4$. The primitive factors satisfy $p \equiv 1 \pmod{2n}$. The identity

$$2^{4i+2} + 1 = (2^{2i+1} - 2^{i+1} + 1)(2^{2i+1} + 2^{i+1} + 1)$$

shows that in this case the factorization of $2^n + 1$ is reduced to factorization of numbers of the form $2^{2i+1} \pm 2^{i+1} + 1$. Four numbers of this type were tried, namely $2^{67} - 2^{34} + 1$, $2^{67} + 2^{34} + 1$, $2^{71} - 2^{36} + 1$, and $2^{73} - 2^{37} + 1$, of which the first two are the factors of $2^{134} + 1$, and the others are factors of $2^{142} + 1$ and $2^{146} + 1$.

In each of the thirty cases, all possible divisors less than 2^{30} were tried. The time for each run was about half an hour, except that when factors were found, this was reduced, in some cases very sharply. The following factorizations were found:

$$
\begin{aligned}
2^{67} - 2^{34} + 1 &= 5 \cdot 269 \cdot 42\,875\,177 \cdot 2\,559\,066\,073,\\
2^{67} + 2^{34} + 1 &= 15\,152\,453 \cdot 9\,739\,278\,030\,221,\\
2^{71} + 1 &= 3 \cdot 56\,409\,643 \cdot 13\,952\,598\,148\,481,\\
2^{95} - 1 &= 31 \cdot 191 \cdot 524\,287 \cdot 420\,778\,751 \cdot 30\,327\,152\,671,\\
2^{109} - 1 &= 745\,988\,807 \cdot X,\\
2^{109} + 1 &= 3 \cdot 104\,124\,649 \cdot Y,\\
2^{112} + 1 &= 449 \cdot 2689 \cdot 65\,537 \cdot 183\,076\,097 \cdot 358\,429\,848\,460\,993,\\
2^{113} + 1 &= 3 \cdot 227 \cdot 48\,817 \cdot 636\,190\,001 \cdot 491\,003\,369\,344\,660\,409,\\
2^{157} - 1 &= 852\,133\,201 \cdot Z.
\end{aligned}
$$

All the factors written out are less than 2^{60}, and hence are shown to be prime by the routine. This fact was checked by an additional run. The three factors X, Y, Z exceed 2^{60}, and are of an unknown character.

Thus we have completely factored the numbers $2^{71} + 1$, $2^{95} - 1$, $2^{112} + 1$, $2^{113} + 1$, and $2^{134} + 1$, the last factorization being obtained by multiplying together the first two of the above equations. (The entry for $2^{134} + 1$ in Cunningham and Woodall [1] erroneously gives the product of the two largest prime factors of $2^{67} - 2^{34} + 1$ as a prime.) We have also found factors of two Mersenne numbers, $2^{109} - 1$ and $2^{157} - 1$, for which no factor was previously known, and a factor of $(2^{109} + 1)/3$.

No factors other than those shown above were found in any of the cases tried, except for the algebraic factors and the factors less than 300,000 which appear in Cunningham and Woodall [1]. Check runs were not made, but it appears to be quite unlikely that any factor less than 2^{30} was missed.

Appendix. At the suggestion of the referee, two lists are included which show the progress which has been made in factoring numbers of the form $2^n \pm 1$. These lists have been prepared with the help of D. H. Lehmer and J. L. Selfridge. First we have a list of all the cases we could find in which complete factorizations have been claimed, where for $2^n - 1$ only odd values of n are considered.

$2^n - 1$: n = 1–99, 105, 107, 111, 113, 115, 117, 123, 127, 129, 135, 151, 521, 607, 1279, 2203, 2281

$2^n + 1$: n = 0–102, 105, 106, 108, 110, 111, 112, 113, 114, 118, 120, 122, 123, 126, 130, 134, 135, 138, 146*, 148, 150, 154, 162, 170, 174, 182, 186*, 190*, 198, 210, 234*, 270

Some of these were not known to the author at the time the work described above was carried out. It should be mentioned that in several cases there is doubt that the factorizations are in fact complete; this is true, in particular, in the cases marked with an asterisk. Notice that M. Kraitchik [6] had already given a supposedly complete factorization of $2^{95} - 1$, but that we found above that a further decomposition of one of his factors is possible.

The second list concerns the "original Mersenne numbers" (that is, numbers of the form $2^p - 1$ where p is prime and $p \leq 257$), and brings up to date a similar list by R. C. Archibald in *MTAC* [7].

p	Character of $2^p - 1$
2,3,5,7,13,17,19,31,61,89,107,127	Prime
(All other $p < 100$),113,151	Composite and completely factored
163,173,179,181,223,233,239,251	Two or more prime factors known
109,131,157,167,191,197,211,229	Only one prime factor known
101,103,137,139,149,193,199,227,241,257	Composite but no factor known

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The operation of this computer is supported in part by the National Science Foundation.

1. A. J. C. CUNNINGHAM & H. J. WOODALL, *Factorisation of* $y^n \mp 1$, $y = 2, 3, 5, 6, 7, 10, 11, 12$ *up to high powers* (n), Francis Hodgson, London, 1925.
2. D. H. LEHMER, "On the factors of $2^n \pm 1$," Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 164–167.
3. RAPHAEL M. ROBINSON, "Mersenne and Fermat numbers," Amer. Math. Soc., *Proc.*, v. 5, 1954, p. 842–846.
4. RAPHAEL M. ROBINSON, "Factors of Fermat numbers," *MTAC*, v. 11, 1957, p. 21–22.
5. J. L. SELFRIDGE, "Factors of Fermat numbers," *MTAC*, v. 7, 1953, p. 274–275.
6. M. KRAITCHIK, *Introduction à la Théorie des Nombres*, Gauthier-Villars, Paris, 1952. p. 39.
7. R. C. ARCHIBALD, "Mersenne numbers," *MTAC*, v. 3, Note 98, 1949, p. 398.

On the Solution of "Jury" Problems with Many Degrees of Freedom

1. Introduction. In a recent numerical investigation using the Differential Analyser, it was found necessary to solve differential equations of up to the eighth order with two-point boundary conditions, the so-called "Jury" problem. Now,

for second order equations with only one degree of freedom various processes of inverse interpolation may be used to estimate, from trial solutions, the correct initial conditions for the required solution. Of these, perhaps the best for this purpose is that due to Aitken [1, 2], which is, as it stands, equally suitable for direct or inverse interpolation. An example of its application is given in section 2.

The method given below is an extension to many variables of a similar process. If we have n functions $f(u, v, \cdots, z)$, $g(u, v, \cdots, z)$, $\cdots$, $m(u, v, \cdots, z)$, of the same n independent variables, it is necessary to interpolate between $n + 1$ sets of values of $f, g, \cdots, m$, to find the n values of $u, v, \cdots, z$, required to form the wanted values of the functions. With more than one variable we are restricted to linear interpolates only, as the extension to higher order interpolates turns out to be unprofitable and difficult to generalise. It is found to be simpler to use the process of linear inverse interpolation iteratively, obtaining from the first $n + 1$ sets of values a further set based on the first linear interpolation, then using this and the best n preceding sets to repeat the process.

2. Aitken's process for one variable. This may be illustrated by means of an example of a "Jury" problem. Consider the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x^2 = 0,$$

where the solution is required satisfying the boundary conditions

$$x(u) = a,\ x(v) = b.$$

We may then integrate forward from $t = u$ with two arbitrary initial values of x', say x_1' and x_2', to reach $t = v$, when $x = c_1$ and c_2 respectively. Defining $d = c - b$, we may interpolate $d = 0$ to find, in the usual notation,

$$x_3' = x'(0; d_1, d_2)$$

as a starting value of x' for a new run. This produces d_3, and further interpolation yields *both* $x'(0; d_1, d_3)$ and $x'(0; d_1, d_2, d_3)$. The sequence of operations may be shown as:

$$\begin{array}{llllll}
x_1' & c_1 & d_1 & & & \\
x_2' & c_2 & d_2 & x'(0; d_1, d_2) = x_3' & & \\
x_3' & c_3 & d_3 & x'(0; d_1, d_3) & x'(0; d_1, d_2, d_3) = x_4' & \\
x_4' & c_4 & d_4 & x'(0; d_1, d_4) & x'(0; d_1, d_2, d_4) & x'(0; d_1, d_2, d_3, d_4) = x_5' \\
x_5' & & & & &
\end{array}$$

and so on.

It should be noted in passing that convergence is not guaranteed, since a solution of the type we want may not exist. Results for the above equation with the boundary conditions $x(0) = 1$, $x(1) = 0$ are shown in Table 1.

TABLE 1

run	x' initially	x finally				parts
1	−1	+0.1875				+1875
2	−1.2	+0.0811	−1.3524			+ 811
3	−1.3524	−0.0015	−1.3496	−1.3497		− 15
4	−1.3497	−0.0003	−1.3491	−1.3491	−1.3490	− 3
5	−1.3490	0	—	—	—	

3. Extension to many variables. Let us consider the case of three variables (three degrees of freedom) for simplicity in writing. The extension to the general case from this is immediate and obvious.

Then we have three parameters u, v, w, whose values at the start of the range have to be estimated so as to make three functions f, g, h of these parameters zero at the end of the range.

Now $f = f(u, v, w)$, $g = g(u, v, w)$, $h = h(u, v, w)$ may be inverted to give $u = u(f, g, h)$, $v = v(f, g, h)$, $w = w(f, g, h)$, provided that the Jacobian of the transformation does not vanish at any of the points in question. This we may assume for any given set of numerical functions.

Denoting the values of $u, v, w, \dfrac{\partial u}{\partial f}, \cdots, \dfrac{\partial w}{\partial h}$ taken when $f = g = h = 0$ by $U, V, W, U_f, \cdots, W_h$, respectively, then we obtain from Taylor's series truncated after its first-order terms

$$u = U + fU_f + gU_g + hU_h.$$

From four such equations (derived from the approximations u_1, u_2, u_3, u_4)

$$f_1U_f + g_1U_g + h_1U_h + U = u_1,$$

$$f_2U_f + g_2U_g + h_2U_h + U = u_2,$$

$$f_3U_f + g_3U_g + h_3U_h + U = u_3,$$

$$f_4U_f + g_4U_g + h_4U_h + U = u_4,$$

we may eliminate U_f, U_g, U_h to obtain the required interpolate U. This set of equations may be solved by the Gauss elimination method [3] to give U directly, with no back-substitution.

Further, the sets of equations for V and W differ from the above set for U only in that u_1 to u_4 are replaced by v_1 to v_4 and w_1 to w_4 respectively. Hence the whole process may be represented in matrix form as

$$\begin{bmatrix} f_1 & g_1 & h_1 & 1 \\ f_2 & g_2 & h_2 & 1 \\ f_3 & g_3 & h_3 & 1 \\ f_4 & g_4 & h_4 & 1 \end{bmatrix} \begin{bmatrix} U_f & V_f & W_f \\ U_g & V_g & W_g \\ U_h & V_h & W_h \\ U & V & W \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{bmatrix},$$

and the elimination process may be carried through to give a result of the form

$$[FU \quad FV \quad FW] = [\bar{u} \quad \bar{v} \quad \bar{w}],$$

whence $U = \bar{u}/F$, $V = \bar{v}/F$, $W = \bar{w}/F$. This process may then be repeated with four sets of values comprising (U, V, W) and the three best approximations from $(u_1, v_1, w_1), \cdots, (u_4, v_4, w_4)$. There is a corresponding reduction in labour if the asymmetrical Cholesky process [3] is used.

Formal solution of the matrices gives

$$U = \frac{| u_1 f_2 g_3 h_4 |}{| 1 f_2 g_3 h_4 |},$$

which is seen to be a (formal) extension of the linear interpolation formula in the Aitken process, since the usual formula

$$f(u; f_1, f_2) = (u_1 f_2 - u_2 f_1)/(f_2 - f_1)$$

can be rewritten

$$f(u; f_1, f_2) = \frac{| u_1 f_2 |}{| 1 f_2 |}.$$

4. Example. As an example of this process, consider the steady-state solution of van der Pol's equation subject to a forcing function,

$$\frac{d^2x}{dt^2} - (1 - x^2)\frac{dx}{dt} + x = \cos t.$$

If $x(0) = a$, $x'(0) = b$, we may be required to estimate both a and b so that $x(2\pi) = a$, $x'(2\pi) = b$, that is, the solution has the same period as the forcing function. For a given (a, b) let $x(2\pi) = c$, $x'(2\pi) = d$, and define $f = a - c$, $g = b - d$. Then we may integrate forward with three independent sets of (a, b) and then interpolate inversely to estimate values $\hat{a}$, $\hat{b}$ corresponding to $f = g = 0$. The results are shown in Table 2, and exhibit quite a rapid convergence.

TABLE 2

run	*a*	*b*	*c*	*d*	*f*	*g*	$\hat{a}$	$\hat{b}$	*Computed using runs*
1	+1	0	+1.227	−1.411	−0.227	+1.411	—	—	—
2	+1	−1	+1.140	−1.499	−0.140	+0.499	—	—	—
3	+1.200	−1.600	+1.196	−1.443	+0.004	−0.157	+1.2271	−1.4106	1,2,3
4	+1.227	−1.411	+1.197	−1.441	+0.030	+0.030	+1.1903	−1.4482	2,3,4
5	+1.190	−1.448	+1.190	−1.448	—	—	—	—	—

5. Conclusion. The method described above has been used on various differential equations and has shown quite good convergence. When a series of solutions with varying parameters have to be found, it is usually possible to "predict" a set of values lying near the required ones. The convergence is then more rapid.

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1. L. M. MILNE-THOMSON, *The Calculus of Finite Differences*, Macmillan, London, 1933, § 3.81.

2. M. C. K. TWEEDIE, "A modification of the Aitken-Neville Linear iterative procedures for polynomial interpolation," *MTAC*, v. 8, 1954, p. 13–16.

3. L. FOX, "Practical solution of linear equations and inversion of matrices," NBS Applied Mathematics Series, v. 39, 1954, p. 1–54.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

106[C, D].—JUAN GARCIA, "Nuevas Tablas de Logaritmos," *Las Ciencias*, Madrid, Spain, v. 19, 1954, p. 567–592.

A new, five-place "triple entry" table of 5D logarithms of numbers and of circular functions is described with the aid of representative excerpts from the table. Instructions for its use are in the form of varied and illustrative examples.

With regard to the logarithmic tabulation of the angles in the intervals (0°, 3°) and (87°, 90°), the table provides a direct and simple interpolation scheme for such angles; the arrangement is "triple entry."

Uniformity in the methods of tabulation and interpolation for the angles throughout the interval (0°, 90°) is attained. The table is compact, and its entries can be read simply and efficiently.

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107[F].—D. N. LEHMER, *Factor Table for the First Ten Millions Containing the Smallest Factor of Every Number not Divisible by 2, 3, 5, or 7 between the Limits 0 and 10017000*, Hafner Publishing Co., New York, 1956, 9 p. in double col. + 476 p., 43 cm. × 30 cm., oblong. Price $22.50. Originally issued as Carnegie Institution of Washington Publication No. 105, 1909.

[F].—D. N. LEHMER, *List of Prime Numbers from 1 to 10006721*, Hafner Publishing Co., New York, 1956, 8 p. in double cols. + 133 p., 43 cm × 30 cm., oblong. Price $15.00. Originally issued as Carnegie Institution of Washington Publication No. 165, 1914.

The availability of these two monumental tables is a cause for rejoicing among all devotees of the theory of numbers—and only a very few of us are not. The author begins his preparatory note to the first of these volumes, "Factors," with the sentence: "The value of a factor table depends chiefly on its freedom from errors." Time has certainly proved the value of this book for no errors have been reported in it; only two serious errors have been found in the companion volume, "Primes." They are listed here.

p. 11, col. 13, line 1, *for* 8151 *read* 8051.
p. 14, col. 30, line 55, *for* 51 *read* 47.
p. 99, col. 20, heading, *for* 224 *read* 724.
p. 119, col. 25, heading, *for* 83 *read* 883.

In the review copy the first of these errors has been corrected while the second has been marked but not corrected. The first three were noted by D. H. Lehmer [1]. The last was pointed out by E. G. H. Comfort (Ripon College, Wisconsin).

Professor Comfort has also pointed out that the description of the Kulik tables in the "Primes" need to be revised in the light of the note by S. A. Joffe [2]. He also noted that he received a copy which had pages 68 and 69 replaced by 28 and 29 and pages 72 and 73 replaced by 48 and 49. The review copy appears faultless in this respect.

Both books have been reproduced by photographic methods from copies of the original edition—which were themselves reproduced photographically from typescript. The original editions were some of the earliest major tables to be produced in this manner.

For those who are not familiar with the tables we give brief descriptions. "Factors" gives the least prime factor of each $n \leq 10017000$, provided n is not divisible by 2, 3, 5, 7. The most convenient way of using it is to find q, r such that $n = 210q + r$, $0 \leq r < 210$ (using a desk calculator, for instance). Then, turning to the page given by the first three digits of q and the line given by the last two digits, we read in the column with heading r, the least prime factor of n, or a dash indicating that n is prime. If r is not among the column headings, a factor 2, 3, 5, or 7 is present and must be removed before using the table. Repeated entry into the table gives the complete factorization of n. By means of this a much greater range can be covered than the other modern tables BAAS [3] which give complete factorization.

In "Primes" we find a listing, 5000 to a page, of all the primes up to 10006721.

Both volumes contain careful introductions which describe the construction of the tables and their checking and printing. In "Primes" there are tables giving comparisons of $\pi(x)$, the number of primes $\leq x$, with various approximate formulae. In "Factors" there is a list of errors discovered in earlier tables.

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1. D. H. Lehmer, *Guide to Tables in the Theory of Numbers*, Bull. National Research Council 105, 1941, p. 161.
2. S. A. Joffe, Editorial Note in connection with Kulik's *Magnus Canon Divisorum* . . . , UMT **48**, *MTAC*, v. 2, 1946, p. 139–140.
3. J. Peters, A. Lodge, E. J. Ternouth, & E. Gifford, *Factor Table giving the Complete Decomposition of all Numbers less than 100,000*. (British Association for the Advancement of Science, *Mathematical Tables*, v. 5.) London, BAAS, 1935.
4. J. Kaván, *Factor Tables giving the Complete Decomposition into Prime Factors of All Numbers up to 256,000* . . . , Macmillan, London, 1937. [RMT **196**, *MTAC*, v. 1, 1945, p. 420–421.]

108[F, L].—D. H. Lehmer, "Extended computation of the Riemann zeta-function," *Mathematika*, v. 3, 1956, p. 102–108.

This paper describes the methods used by the author in the computation of the first 25,000 zeros of the Riemann zeta-function $\zeta(s)$ which was carried out on SWAC. All these zeros have Re $s = \frac{1}{2}$.

A table of the coefficients of 4 polynomials entering in the calculation is included, and there is a table of the number of failures of Gram's law divided into 3 types and tabulated in 15 sets of 1000 roots each, beginning with root number 9892, which overlaps the previous run [1].

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1. D. H. Lehmer, "On the roots of the Riemann zeta-function," *Acta Mathematica*, v. 95, 1956, p. 291–298 [*MTAC*, Review **52**, v. 11, 1957, p. 107–8].

109[F].—ALBERT GLODEN, *Table de factorisation des nombres* $N^4 + 1$ *dans l'intervalle* $3000 < N \leq 6000$, published by the author, rue Jean Jaurès, 11, Luxembourg, 25 p., 30 cm., mimeographed. Price 125 francs belges. A copy deposited in the UMT FILE.

This table is a revision of UMT **108** [1] with all unknown factors $> 8.10^5$. The author intends to extend the tables [2] of solutions of $x^4 + 1 \equiv 0 \pmod{p}$ to 10^6. The UMT files also contain an earlier revision of UMT **108** published in 1952.

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1. A. GLODEN, *Table de factorisation des nombres* $N^4 + 1$ *dans l'intervalle* 3001–6000. [UMT **108**, *MTAC*, v. 4, 1950, p. 224.]
2. S. HOPPENOT, *Table des Solutions de la Congruence* $x^4 \equiv -1$ *(mod N) pour* $100000 < N < 200000$, Brussels, Librairie du "Sphinx," 1935. [RMT **48**, *MTAC*, v. 1, 1943, p. 6.]
3. A. GLODEN, "Table des solutions de la congruence $X^4 + 1 \equiv 0 \pmod{p}$ pour $2.10^5 < p < 3.10^5$," *Mathematica* (Rumania), v. 21, 1945. [RMT **280**, *MTAC*, v. 2, 1946, p. 71–72.]
4. ALBERT DELFIELD, "Table des solutions de la congruence $X^4 + 1 \equiv 0 \pmod{p}$ pour $300000 < p < 350000$," Institut Grand-ducal Luxembourg, Section des Sciences, *Archives*, v. 16, 1946. [RMT **346**, *MTAC*, v. 2, 1947, p. 210–211.]
5. ALBERT GLODEN, *Table des solutions de la congruence* $x^4 + 1 \equiv 0$ *(mod p) pour* $350.000 < p < 500.000$, Luxembourg, author, 11 rue Jean Jaurès, and Paris, Centre de Documentation Universitaire, 1946. [RMT **410**, *MTAC*, v. 2, 1947, p. 300–301.]
6. ALBERT GLODEN, *Table des Solutions de la Congruence* $x^4 + 1 \equiv 0$ *(mod p) pour* $500\,000 > p < 600\,000$, Luxembourg, author, rue Jean Jaurès 11, 1947. [RMT **491**, *MTAC*, v. 3, 1948, p. 96.]
7. A. GLODEN, *Solutions of* $x^4 + 1 \equiv 0$ *(mod p) for* $600000 < p < 800000$ [RMT **1169**, *MTAC*, v. 8, 1954, p. 77.]

110[F].—JOHN LEECH, "Table of groups of 4, 5, 6, 7 primes from 50 to 100 17000," 1 sheet (photostat), 34 × 23 cm., deposited in UMT FILE.

The table gives $15n$ where four primes between 50 and 100 17000 are generated by adding and subtracting 2 and 4 from $15n$. Those values for which additional primes are generated by addition and subtraction of 8 and of these the values which give additional primes by addition and subtraction of 14 and then of 16 are suitably marked. Double fours, primes of the form $210n \pm (11, 13, 17, 19)$ are also marked.

From author's remarks

111[F].—R. M. ROBINSON, "Table of factors of numbers one unit larger than small multiples of powers of two," Los Angeles, 1957, 312 pages listed on an IBM 402 from SWAC punched card output. The table and the punched cards have been deposited in the UMT FILE.

The main table tells whether any number of the form $N = k \cdot 2^n + 1$ is prime or composite for odd $k < 100$ and all $n < 512$. If the least factor of N is less than 10^4 it is listed. For $k = 1, 3, 5, 7$ the bounds on n are larger, namely 2272, 1280, 1536, 1280, and for $n < 2272$ least factors to 10^5 are listed. Each prime found was tested as a possible factor of any Fermat number $F_m = 2^{2^m} + 1$.

In addition to listing the 14 new factors of Fermat numbers previously reported by Robinson [1], the table lists $95 \cdot 2^{61} + 1$ as a factor of F_{58}. This brings to 30 the number of known composite F_m. There are 118 primes greater than 2^{250} in the table, but only the three already mentioned [1] are greater than 2^{508}.

There is a list of the 42 Cullen numbers $n \cdot 2^n + 1$ with $n < 10^3$ for which Cunningham [2] did not find a small factor. These were all tested, and only the smallest, $141 \cdot 2^{141} + 1$, is a prime.

Robinson has presented this wealth of information in a very careful, pleasing, and useful format which is described in the four-page introduction. This was achieved despite the poor type font and limitations of the IBM 402.

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1. R. M. ROBINSON, "Factors of Fermat numbers," *MTAC*, v. 11, 1957, p. 21–22.
2. A. CUNNINGHAM & H. J. WOODALL, "Factorisation of $Q = (2^q \mp q)$ and $(q \cdot 2^q \mp 1)$," *Messenger Math.*, v. 47, 1917, p. 1–38.

112[F].—EMMA LEHMER, "On the location of Gauss sums," *MTAC*, 1956, v. 10, p. 194–202.

The generalized Gauss sum of order k is defined by $S_k = \sum_{m=0}^{p-1} \exp(2\pi i m^k/p)$, $p = kf + 1$ and prime.

It is known that $-(k-1)\sqrt{p} \leq S_k \leq (k-1)\sqrt{p}$. The author investigates the distribution of S_k for $k = 3, 4, 5, 7$ with respect to various partitions of the interval $(-(k-1)\sqrt{p}, (k-1)\sqrt{p})$. The research was undertaken to test Kummer's conjecture which states that S_3 falls in the intervals $(-2\sqrt{p}, -\sqrt{p})$, $(-\sqrt{p}, \sqrt{p})$, $(\sqrt{p}, 2\sqrt{p})$ with frequencies of 1 to 2 to 3. It was found that for the first 1000 primes of the form $6n + 1$, these ratios became 3 to 4 to 5, tending to disprove the conjecture.

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113[F].—R. J. PORTER, *Irregular Negative Determinants of Exponent 3n with their Critical Classes. Part II, from* $-D = 50{,}000$ *to* 100,000. 203 typewritten pages, 25.5 × 10 cm., deposited in UMT FILE.

This table is a supplement to *Table of Irregular Negative Determinants of Exponent 3n up to* $-D = 50{,}000$, which were previously deposited in the UMT FILE [*MTAC*, Review **3**, v. 9, 1955, p. 26 and *MTAC*, Review **84**, v. 9, 1955, Part I, p. 198].

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114[G, X].—D. H. LEHMER, "On certain character matrices, "*Pacific J. Math.*, v. 6, 1956, p. 491–499.

Two classes of matrices M, for which det M, $\lambda_i(M)$, $(M^{-1})_{ij}$ (if existing) and $(M^k)_{ij}$ can be given explicitly, are described. (Here $\lambda_i(M)$ denotes the characteristic roots of M.) The matrices are of order $p - 1$, where p is an odd prime, and they are defined in terms of the Legendre symbol

$$\chi(n) = \left(\frac{n}{p}\right) = \begin{cases} 0 \text{ if } p \text{ divides } n \\ -1 \text{ if the congruence } x^2 \equiv n \pmod{p} \text{ is impossible.} \\ +1 \text{ otherwise} \end{cases}$$

The general element of a matrix in the first class is

$$M_{ij} = a + b\chi(i) + c\chi(j) + d\chi(ij)$$

for any a, b, c, d. This is singular except for $p = 3$. The general element of a matrix in the second class is

$$M_{ij} = c + \chi(\alpha + i + j)$$

where c is any constant and α any integer.

These matrices are, incidentally, useful for the evaluation of programs for inverting and finding the characteristic roots of matrices on computers.

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115[L].—S. L. BELOUSOV, *Tablit͡sy normirovannykh͡ prisoedinennykh͡ polinomov Lez͡handra* (*Tables of normalized associated Legendre polynomials*). Akad. Nauk SSSR, Energetic͡heskiĭ Institut im G. M. Krz͡hiz͡hanovskogo (Academy of Sciences of the USSR, Krz͡hiz͡hanovskiĭ Institute of Power). Moscow, Press of the Academy of Sciences, 1956. 380 p., 27 cm. Price 23.70 rubles.

These are tables of the associated functions $\bar{P}_n^m$ (cos θ) defined by

$$\bar{P}_n^m (\cos \theta) = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}}\, P_n^m (\cos \theta),$$

the normalization resulting from this definition being such that

$$\int_{-1}^{+1} [\bar{P}_n^m(x)]^2 dx = 1.$$

The functions tabulated are thus identical in every way with those whose behavior is exhibited graphically up to $n = 8$ in four diagrams in the second and subsequent editions of Jahnke and Emde [1]. Except for the difference of symbolism, they are also those denoted by F_n^m (cos θ) in Section 16 of FMR *Index*.

The arguments in the Belousov tables are $m = 0(1)36$, $n = m(1)56$, $\theta = 0(2° .5)90°$. Each pair of values of m, n belongs to one column of 37 entries corresponding to the 37 values of θ. The function values are to 6*D*, without differences. Nine figures were kept in the computations.

The author refers not only to several well-known tables, but also to tables by I͡a. M. Kh͡eĭfet͡s [2], which are stated to give values of associated functions for $m = 1(1)12$, $n = m(1)20$, and each 5° of θ. On page 15 are given nine corrections (two of them taken from Kh͡eĭfet͡s) to numerical values contained in R. and L. Egersdörfer [3].

A. F.

1. E. JAHNKE & F. EMDE, *Funktionentafeln mit Formeln und Kurven*, second edition, Leipzig and Berlin, 1933, p. 178–179; also with various page numbers in later editions.

2. I͡A. M. KH͡EĬFET͡S, *Tablit͡sy normirovannykh͡ prisoedinennykh͡ polinomov Lez͡handra* (*Tables of normalized associated Legendre polynomials*), Moscow, Gidrometeoizdat, 1950.

3. R. and L. EGERSDÖRFER, *Formeln und Tabellen der zugeordneten Kugelfunktionen 1. Art von n = 1 bis n = 20. I. Teil: Formeln.* Reichamt für Wetterdienst, *Wissenschaftliche Abhandlungen*, Bd. I, Nt. 6, Berlin, 1936.

116[L].—George C. Clark & Stuart W. Churchill, *Tables of Legendre Polynomials P_n (cos θ) for $n = 0(1)80$ and $\theta = 0°(1°)180°$*. Engineering Research Institute Publications, University of Michigan Press, Ann Arbor, Mich., 1957, ix + 92 p., 28 cm. Price $4.50.

This is a straightforward table of the ordinary Legendre polynomials $P_n(\cos\theta)$. As the values $P_0(\cos\theta) = 1$ and $P_n(\cos 0) = 1$ are not listed, the tabular arguments are $n = 1(1)80$ and $\theta = 1°(1°)180°$, the limit 180° rather than the usual 90° having been chosen for convenience in application. The function values are to $6D$, and are believed accurate to the last place given, since they were calculated to $11D$ on the Michigan Digital Automatic Computer (MIDAC). No differences are provided. Cosines required in the computations were taken from the NBS 15-place tables.

The table, whose special virtue is the high upper limit of n, was computed in connection with a research program on the engineering applications of light-scattering which has been in progress in the Department of Chemical and Metallurgical Engineering of the University of Michigan for the past ten years.

A. F.

117[L].—Harold K. Crowder & George C. Francis, *Tables of Spherical Bessel Functions and Ordinary Bessel Functions of Order Half an Odd Integer of the First and Second Kinds*. Ballistic Research Laboratories, Memorandum Report No. 1027, Aberdeen Proving Ground, Maryland, 1956, 86 p., multilithed, 8½″ × 11″. Two copies deposited in UMT File.

This gives tables of the functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x), \qquad y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+\frac{1}{2}}(x)$$

and

$$J_{n+\frac{1}{2}}(x), \qquad Y_{n+\frac{1}{2}}(x)$$

for $x = 1(1)50$, $n = 0(1)N$, in all four cases, where N is such that

$$Y_{n+\frac{1}{2}}(x) < 10^{10} < Y_{n+\frac{1}{2}}(x).$$

Some typical values of N are

x	1	10	20	30	40	50
N	10	30	45	59	72	84.

Values are given to 9 decimals for $n < x$ and to 7 significant figures for $n \geq x$.

These tables only partially overlap earlier tables from which the functions can be derived. We mention a few, extracted from material to appear in the projected second edition of the FMR *Index of Mathematical Tables* [1].

Bessel Functions $J_{n+\frac{1}{2}}(x)$

12 decimals, $n = \mu + \frac{1}{2}(1)M + \frac{1}{2}$, $x = 1(1)20$,

with $|J_{\mu-\frac{1}{2}}(x)| > 1$, $|J_{m+\frac{1}{2}}(x)| < \frac{1}{2}\cdot 10^{-12}$

for each x, in the BAAS *Report* for 1925, page 221; this table was prepared by J. R. Airey [2].

$$6 \text{ decimals}, n = -6\tfrac{1}{2}(1)34\tfrac{1}{2}, x = 1(1)20$$

$$n = -6\tfrac{1}{2}(1) + 6\tfrac{1}{2}, x = 21(1)30.$$

This table is by E. C. J. von Lommel [3], and was partially reproduced (up to $n = 18\tfrac{1}{2}$) in G. N. Watson's *Treatise on Bessel Functions* [4]. It was similarly reproduced, to 4 decimals only, in the various editions of *Funktionentafeln mit Formeln und Kurven* by E. Jahnke and F. Emde [5].

A more recent tabulation by K. Reitz [6] gives 5 decimals or figures for

$$n = -30\tfrac{1}{2}(1) + 31\tfrac{1}{2}, \qquad x = 0(0{\cdot}2)20.$$

Stokes's Functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x), \qquad y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+\frac{1}{2}}(x) = (-)^{n+1} j_{-n-1}(x).$$

The two volumes produced by the National Bureau of Standards Computation Laboratory [7] give 8- to 10-figure values, with central second or second and fourth differences for interpolation for $n = -22(1) + 21$ with

$$x = 0(.01)10(.1)25,$$

in general, and for $n = -31(1) - 23, 22(1)30$ with $x = 10(.1)25$. The auxiliary function $\Lambda_m(x) = (\tfrac{1}{2}x)^m m! J_m(x)$ is also tabulated for

$$m = \frac{61}{2}(1) - \frac{29}{2}, \qquad \frac{1}{2}(1)\frac{61}{2}, \qquad x = 0(.1)25$$

with from 7 figures to 9 decimals, with central x-wise differences.

Riccati Bessel Functions

$$xj_n(x), \qquad (-)^n xy_n(x) = -xj_{-n-1}(x).$$

Here we mention only the book of tables by R. O. Gumprecht and C. M. Sliepcevich [8]. This gives values of the first function to 6 decimals and of the second to 6 or 7 figures, in each case with the first derivatives, for

$$n = 0(1)n + d, \qquad x = 1(1)6(2)10(5)100(10)200(50)400$$

where d increases fairly steadily from 4 at $x = 1$ to 20 at $x = 400$. There is also another inconvenient table giving $xj_n(x)$ and its derivative only for very miscellaneous arguments x from $1{\cdot}2$ to 640, and for $n = 0(1)N$, where N varies erratically from 6 near $x = 2$ to 436 at $n = 640$, usually $\tfrac{2}{3}x < N < x + 10$ approximately.

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1. A. Fletcher, J. C. P. Miller, & L. Rosenhead, *An Index of Mathematical Tables*, Scientific Computing Service Ltd., London, 1946.
2. BAAS, *Mathematical Tables Report*, "Bessel functions of half-odd integral order," Cambridge University Press, 1925, p. 221–233.
3. E. C. J. von Lommel, Munich Abhandlungen, v. 15, 1886, p. 529–664.

4. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, New York, 1922.
5. EUGENE JAHNKE & FRITZ EMDE, *Funktionentafeln mit Formeln und Kurven*, Teubner, Leipzig and Berlin, 1909 and 1948.
6. K. REITZ, Institut fur Praktische Mathematik, Darmstadt, Reports, "Tabellierung Besselscher Funktionen," 4. Bericht, p. 2–9; 6. Bericht, p. 2–31; 9 Bericht, p. 2–37, 1945.
7. NATIONAL BUREAU OF STANDARDS COMPUTATION LABORATORY, *Tables of Spherical Bessel Functions*, v. I and II, Columbia University Press, New York, 1947.
8. R. O. GUMPRECHT & C. M. SLIEPCEVICH, *Tables of Riccati Bessel Functions for Large Arguments and Orders*, Engineering Research Institute, University of Michigan, Ann Arbor, 1951.

118[L].—FRANK E. HARRIS, "Tables of the exponential integral Ei(x)," *MTAC*, v. 11, 1957, p. 9–16.

This paper gives basic values of high precision for use in molecular structure calculations, with special reference to the region $4 < |x| < 50$ in which $|x|$ is inconveniently large for the Taylor series and too small for the asymptotic formula.

Table 1. Values of $-\mathrm{Ei}(-x)$ to 18S and of $-e^x\mathrm{Ei}(-x)$ to 19D for $x = 1(1)4(.4)8(1)50$.

Table 2. Values of $\mathrm{Ei}(x)$ to 18S and of $e^{-x}\mathrm{Ei}(x)$ to 19D for the same values of x as in Table 1.

Table 3. Tables of the functions

$$R_n(h) = n!\left[1 - \left(1 + h + \frac{h^2}{2!} + \cdots + \frac{h^n}{n!}\right)e^{-h}\right]$$

which occur in a special interpolation formula. Values of $R_n(h)$ are listed for $h = .01(.01).05, .1(.1).5, 1$ to as many decimal places and up to as large values of n as are necessary.

Table 4. Constants, namely Euler's constant γ and the modulus $\log_{10}e$ to 24D, and $e^{\pm h}$ to 25D for the same values of h as in Table 3.

A. F.

119[L, S].—R. B. DINGLE, D. ARNDT, & S. K. ROY, "The integrals $A_p(x) = (p!)^{-1}\int_0^\infty \epsilon^p(\epsilon + x)^{-1}e^{-\epsilon}d\epsilon$ and $B_p(x) = (p!)^{-1}\int_0^\infty \epsilon^p(\epsilon + x)^{-2}e^{-\epsilon}d\epsilon$ and their tabulation," *Appl. Sci. Research B*, v. 6, 1956, p. 144–154.

[L, S]R. B. DINGLE, D. ARNDT, & S. K. ROY, "The integrals

$$C_p(x) = (p!)^{-1}\int_0^\infty \epsilon^p(\epsilon^2 + x^2)^{-1}e^{-\epsilon}d\epsilon$$

and

$$D_p(x) = (p!)^{-1}\int_0^\infty \epsilon^p(\epsilon^2 + x^2)^{-2}e^{-\epsilon}d\epsilon$$

and their tabulation," *Appl. Sci. Research B.*, v. 6, 1956, p. 155–164.

The integrals are tabulated as follows: $\mathfrak{A}_p(x)$ for $p = -.5(.5)4$, $\mathfrak{B}_p(x)$ for $p = 0(.5)4$, $\mathfrak{C}_p(x)$ for $p = -.5(.5)5$, $\mathfrak{D}_p(x)$ for $p = 0(.5)6.5$ all for

$$x = 0(.1)1(.2)2(.5)10(1)20$$

to 4S except for $\mathfrak{D}_p$ when $x \geq 18$, where 3S are given. It is pointed out that values of $\mathfrak{A}_p$ and $\mathfrak{B}_p$ for imaginary arguments can be obtained from the present tables

using relations such as $\mathfrak{A}_p(ix) = (p + 1)\mathfrak{C}_{p+1}(x) - ix\mathfrak{C}_p(x)$. A table for $\mathfrak{A}_p(-x)$ is promised for later publication. Functions are given to facilitate interpolation between orders p for small x. No discussion of interpolation with respect to x is given.

The second paper includes a table of the Fresnel integrals

$$C(x) = \tfrac{1}{2} \int_0^x J_{\frac{1}{2}}(t)dt \text{ and } S(x) = \tfrac{1}{2} \int_0^x J_{\frac{1}{2}}(t)dt \text{ to 12D for } x = 0(1)20.$$

This table was prepared using the representations of $C(x)$ and $S(x)$ as sums of Bessel functions of half-integral order which are tabulated in BAAS [1]. The authors point out an error in this table caused by transposition of digits:

$$\textit{for } J_{39/2}(14) \textit{ read } 0.00437\ 04731\ 08.$$

The reviewer has checked this value.

The tables for $\mathfrak{A}_p$ and $\mathfrak{B}_p$ were computed, using desk machines, by use of recurrence relations such as $p\mathfrak{A}_p + x\mathfrak{A}_{p-1} = 1$ and $\mathfrak{B}_p = x^{-1}\{1 - (p + x)\mathfrak{A}_p\}$ using initial values obtained from $\mathfrak{A}_0(x) = -e^x\mathrm{Ei}(-x)$ and

$$\mathfrak{A}_{-\frac{1}{2}}(x) = (2/x)^{\frac{1}{2}}F[(2x)^{\frac{1}{2}}]$$

where $F(t)$ is the ratio of the tail area of the normal curve to its bounding ordinate at t, which has been tabulated (BAAS [2]). The tables for $\mathfrak{C}_p$ and $\mathfrak{D}_p$ can be obtained similarly from

$$p(p - 1)\mathfrak{C}_p + x^2\mathfrak{C}_{p-2} = 1 \qquad \text{and} \qquad \mathfrak{D}_p = \tfrac{1}{2}x^{-2}\{(p + 1)\mathfrak{C}_{p+1} - (p - 1)\mathfrak{C}_p\}$$

and expressions for $\mathfrak{C}_0$ and $\mathfrak{C}_1$ in terms of $\mathrm{Si}(x) = \int_\infty^x t^{-1} \sin t\, dt$ and $\mathrm{Ci}(x) = \int_\infty^x t^{-1} \cos t\, dt$ and for $\mathfrak{C}_{\pm\frac{1}{2}}(x)$ in terms of $C(x)$ and $S(x)$. The necessity for care in the use of the recurrence relations is pointed out. The authors state, in a letter to the reviewer, that the results were monitored by calculations of asymptotic representations of $\mathfrak{A}_p$ and $\mathfrak{C}_p$ for large p.

The reviewer checked a substantial portion of the tables for $\mathfrak{A}_p$ and $\mathfrak{B}_p$ using SEAC. It is possible to compute $\mathfrak{A}_p$, for integral p, by use of a Laguerre quadrature formula and, for half-integral p, by use of a Hermite quadrature formula after a trivial change of variable. We can handle $\mathfrak{B}_p$, $\mathfrak{C}_p$, and $\mathfrak{D}_p$ analogously using decompositions into complex partial fractions. The error E in the use of these formulas can be expressed in terms of the divided differences of functions such as $y^p/(y + x)$. For example, for $\mathfrak{A}_p(p \geq 0)$, one has

$$|E| < x^p(m!)^2 \Big/ \left[\prod_{i=1}^{m} (x + x_i)\right]^2 (x + \xi), \qquad 0 < \xi < \infty$$

where x_i is the i-th zero of the Laguerre polynomial of degree m. Thus, e.g., for $x = 2$, $m = 15$, $p = 4$ a crude estimate of the error is 4×10^{-8}. As a sample of the accuracy actually obtained, agreement with the present tables was achieved for $x \geq .7$ in the cases $p = 0$ and $p = 1$.

In addition to the tables, the papers treat the following topics in connection with the functions $\mathfrak{A}_p$, $\mathfrak{B}_p$, $\mathfrak{C}_p$ and $\mathfrak{D}_p$: physical applications (in particular, their origin in the theory of elemental semiconductors), recurrence relations, related

differential equations, relationships to other special functions, and asymptotic expansions.

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1. BAAS, *Mathematical Tables Report*, Cambridge University Press, 1925, p. 221–233.
2. BAAS, *Mathematical Tables*, v. 7, *The Probability Integral*, W. F. Sheppard, Cambridge University Press, 1939.

120[L, S].—M. DANK & S. W. BARBER, "The specific heat function for a two-dimensional continuum. Numerical values of

$$\frac{C_2}{C_\infty} = \frac{6}{x^2}\int_0^x \frac{\xi^2 d\xi}{e^\xi - 1} - \frac{2x}{e^x - 1}."$$

[*MTAC*, v. 9, 1955, p. 191–194]

This function was computed by using a power series for the range $0 \leq x \leq 2.0$ and by using a different expansion for the range $2.0 \leq x \leq 16.0$. The maximum error using seven terms of the power series is less than 0.5×10^{-8} and for the second expansion it is approximately 2×10^{-6}. The table lists values $x = (0.1)9.4$ and 10(.5)16, 5D.

A. H. T.

121[M].—F. OBERHETTINGER, *Tabellen zur Fourier Transformation*, Springer-Verlag, Berlin, 1957, x + 213 p., 24 cm. Price DM 35.80.

This collection of Fourier transforms contains 109 pages of Fourier cosine transforms, 91 pages of Foureir sine transforms, and 6 pages of exponential Fourier transforms. Also given are concise definitions of the function symbols used, a table of errata, and a short bibliography. The book appears to be the largest collection of Fourier transforms published to date. The author states in the preface that a considerable portion of the approximately 1800 correspondences collected in the volume is new. Fourier transforms of elementary functions occupy about two-fifths of the space; the remainder is taken up by transforms of higher transcendental functions, with the Bessel functions and their associates controlling a strong minority.

Glancing over the book one cannot help being impressed by the enormous wealth of formulas stacked up in the book and by the astounding formal dexterity which Professor Oberhettinger must have commanded in deriving some of them. On the other hand one also notices some shortcomings. There is no indication of the sources of individual formulas, conditions for validity are given in many cases for real values of the parameters only; there is no system of numbering the formulas that could be used for reference purposes; and the bibliography refers only to a few well-known titles. However, one must not forget that all these matters are of secondary interest to most of the actual users of such tables and that the absence of a complicated scholarly apparatus makes for easier reading.

The existence of some trivial printing errors (on p. 210 and 213) does not increase the confidence in the absence of non-trivial errors. With this reservation, the printing of the book is excellent; by its neat and clean type it is distinguished

from the ugly appearance of the Varitype process used in a recent American counterpart. Both author and publisher are to be congratulated on this excellent work.

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122[M].—C. F. LINDMAN, *Examen des Nouvelles Tables d'Intégrales Définies de M. Bierens de Haan*, G. E. Stechert and Co., New York, 1944, 231 p., 27 cm. Price $5.00.

According to the preface, Mr. Lindman has inspected Bierens des Haan's famous tables page by page, checking the integrals against such original sources as were available to him and recomputing a considerable number of the integrals. He collected the results of his inspection in a weighty memoir, which was submitted to the Royal Swedish Academy of Sciences and published in 1891. The volume under review is an unaltered reproduction of this memoir. Although many of Lindman's comments consist in obvious restrictions on the parameters of an integral to insure convergence, there is a surprisingly large number of non-trivial corrections. It appears that the majority of the 486 tables which make up Bierens de Haan's collection contain faulty results.

In our review of the 1957 Hafner reprint of the *Nouvelles Tables d'Intégrales Définies* [1] we pointed out that the extent and nature of the corrections made in that edition (as well as in an earlier 1939 reprint) were not evident. Several spot checks now reveal that the results of Lindman's *Examen* have not been incorporated. Thus the work under review is an indispensable companion volume also to the modern editions of Bierens de Haan's Tables (An earlier review of Lindman's *Examen* appeared in *MTAC* [2]).

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1. D. BIERNES DE HAAN, *Nouvelles Tables d'Intégrales Définies*, Hafner Publishing Co., New York, 1957. [Review 60, *MTAC*, v. 11, 1957, p. 111.]
2. CHRISTIAN FREDRIK LINDMAN (1816–1901), *Examen des Nouvelles Tables d'Intégrales Définies de M. Bierens de Haan*, Amsterdam, [sic] 1867. (K. Svenska Vetenskaps Akad., *Handlingar*, v. 24, no. 5, Stockholm, 1891.) New York, G. E. Stechert & Co., 1944. [Review 167, *MTAC*, v. 1, 1944, p. 321–322.]

123[P].—LEONARD PODE, *Tables for Computing the Equilibrium Configuration of a Flexible Cable in a Uniform Stream*, Report 687, The David W. Taylor Model Basin, Washington, D. C., 1951, 31 + 192 p. of tables, 27 cm. This report will be available from the Photoduplication Service, Library of Congress, Washington 25, D. C.

[P.]—LEONARD PODE & LOUIS ROSENTHAL, *Cable Function Tables for Small Critical Angles*, Supplement to Report 687, The David W. Taylor Model Basin, Washington, D. C., 1955, 105 p., 27 cm.

(A) The equations describing the tension in and the configuration of a flexible cable immersed in a uniform steady stream and lying entirely in a plane are de-

rived. They are

$$\ln \tau = \int_{\varphi_0}^{\varphi} \frac{f \frac{\cos\varphi}{\boxed{\cos\varphi}} + w \sin\varphi}{q(\varphi)} \, d\varphi,$$

$$\sigma = \int_{\varphi_0}^{\varphi} \frac{\tau}{q(\varphi)} \, d\varphi,$$

$$\xi = \int_{\varphi_0}^{\varphi} \frac{\tau \cos\varphi}{q(\varphi)} \, d\varphi,$$

$$\eta = \int_{\varphi_0}^{\varphi} \frac{\tau \sin\varphi}{q(\varphi)} \, d\varphi,$$

where

$$q(\varphi) = -\sin\varphi \mid \sin\varphi \mid + w \cos\varphi.$$

Here ϕ_0 is the value of ϕ at the point chosen as the origin of the coordinate system, F is the drag per unit length of cable when the cable is parallel to the stream, R is the drag per unit length of the cable when the cable is normal to the stream, $f = F/R$, T_0 is the tension in the cable at the point chosen as origin of the coordinate system, T is the tension in the cable at the point x, y, $\tau = T/T_0$, $\xi = Rx/T_0$, and $\eta = Ry/T_0$.

These functions called the cable functions are tabulated to four decimal places in this report.

The values of the parameter f for which the cable functions have been evaluated are 0.01, 0.02 and 0.03. Another important parameter entering into the evaluation of these functions is ϕ_c, the critical angle of the cable, the value of the angle ϕ obtained when the cable is freely trailed in the stream and the angle for $q(\phi) = 0$. The values $\phi_c = 0°(5)$ to $85°$ are covered in the tables. These were computed by use of Simpson's rule for the evaluation of integrals. The relative errors in the functions tabulated are said to be less than 0.001 percent.

(B) The cable function tables for small critical angles reported herein are supplementary to those described above. The supplementary tables provide tables for critical angles in the range from 0 to 10 degrees in increments of one degree. More closely spaced intervals of the independent variable in the vicinity of the critical angle ϕ_c are also provided. Numerical integrations were made using the formula

$$\int_{x_n-2h}^{x_n} y dx = \frac{h}{3} [y_n + 4y_{n-1} + y_{n-2}] - \frac{h}{90} [y_n - 4y_{n-1} + 6y_{n-2} - 4y_{n-3} + y_{n-4}].$$

The authors state that "It is believed that the maximum error in any of the tabulated values is never greater than one unit in the least significant figure."

A. H. T.

124[P, K, X, Z].—E. F. Beckenbach, Editor, *Modern Mathematics for the Engineer*, McGraw-Hill, New York, 1956, xx + 514 p., 23 cm. Price $7.50.

This collection of essays is based on a series of lectures organized in the Extension Division of University of California at Los Angeles and given at that

University and repeated elsewhere. Those who did not hear the speakers will be grateful to them, and the editor, E. F. Beckenbach, for providing this volume. Engineers are some of the greatest customers of the automatic computers and much of the present volume is relevant in the field of this journal, and all our readers can profit from reading those essays.

The essays are divided into three parts: (I) Mathematical Models, (II) Probabilistic Problems, and (III) Computational Considerations. The contributors and titles are: R. Weller, Introduction; S. Lefschetz, Linear and nonlinear oscillations; R. Bellman, Equilibrium analysis: the stability theory of Poincaré and Liapunov; J. W. Green, Exterior ballistics; M. R. Hestenes, Elements of the calculus of variations; R. Courant, Hyperbolic partial differential equations and applications; M. M. Schiffer, Boundary-value problems in elliptic partial differential equations; I. S. Sokolnikoff, The elastostatic boundary-value problem; N. Wiener, The theory of prediction; H. F. Bohnenblust, The theory of games; G. W. King, Applied mathematics in operations research; R. Bellman, The theory of dynamic programming; G. W. Brown, Monte Carlo methods; L. A. Pipes, Matrices in engineering; J. L. Barnes, Functional transformations for engineering design; E. F. Beckenbach, Conformal mapping methods; C. B. Morrey, Jr., Nonlinear methods; G. E. Forsythe, What are relaxation methods?; C. B. Tompkins, Methods of steep descent; D. H. Lehmer, High speed computing devices and their applications.

It is not possible to discuss all the contributions; we shall only comment briefly on a few which are specially relevant to the field of this journal.

In his essay in the first part, R. Courant indicates the importance of a double attack on problems, by modern computing techniques and by penetrating analysis. In the second part, the essays by H. F. Bohnenblust and G. W. Brown give an account of subjects which, although they have largely developed in the last few decades, are becoming more and more used in the programming and operational aspects of engineering.

Some of the articles in the third part are not very closely related with practical computational considerations. However, those of Morrey and Forsythe do get down to numerical cases. Morrey discusses the solution of functional equations, mainly arising in calculus of variation problems. Newton's method is discussed in the finite dimensional case, normed linear spaces are introduced and the Rayleigh-Ritz method is discussed. Examples of the solution of a pair of equations in two variables, a first order differential equation and of a calculus of variations problem are discussed in detail.

Forsythe gives a valuable introduction to the subject of relaxation methods, indicating the scope of the method and its extensions, giving illuminating examples, and discussing the basic questions of convergence which arise. There is an ample bibliography.

Tompkins discusses the method of steep descent in a general framework: the solution of many engineering and physical problems can be represented as minimizing some function of several variables, or some integral. There are accounts of the solution of a system of linear equations by the projection method of Kaczmarz and the conjugate gradient method of Hestenes and Stiefel. After appropriate generalizations, a representative problem of the calculus of variations

is discussed and some account is given of recent work in this field which is relevant in connection with computational problems.

The concluding essay by D. H. Lehmer is a pleasant introduction to automatic computers and includes many significant remarks. After noting the inadequacy of current machines for handling differential equations involving three space variables and time he adds, "It is only fair to state that there are also inadequacies in the numerical analysis of such problems." He encourages engineers to do their own programming and coding—a list of the machine instructions is almost enough —and most of the tricks of coding will occur to an engineer "who uses his normal budget of native cunning."

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125[S].—A. RAHMAN, *Tables of Integrals* $A_n(\alpha) = \int_1^\infty \lambda^n e^{-\alpha\lambda} d\lambda$ *and* $F_n(\alpha) = \int_1^\infty Q_0(\lambda)\lambda^n e^{-\alpha\lambda} d\lambda$, Extrait des *Annales de la Societe Scientifique*, T. LXIX, Serie I, 1955, p. 123–128.

This paper gives the functions mentioned in the title $\left(\text{with } Q_0(\lambda) = \frac{1}{2}\ln\frac{\lambda+1}{\lambda-1}\right)$ to ten significant figures for $n = 0(1)10$, $\alpha = .1(.1)3$; 10S. The computations were carried out to 12 significant figures. Interpolation formulae are given. The functions here tabulated are used in calculating the wave functions for the hydrogen molecule.

A. H. T.

126[S].—K. M. CASE, F. DE HOFFMAN, & G. PLACZEK, *Introduction to the Theory of Neutron Diffusion*, v. 1, 1953, Los Alamos Scientific Lab., Los Alamos, New Mexico, viii + 174 p., 26 cm. U. S. Gov. Printing Office, Washington, D. C., Price $1.25.

This book discusses the determination of the distribution of neutrons in space and time in terms of the geometrical configuration and physical properties of the medium they are in under the restrictive assumption that the magnitude of the neutron velocity is unchanged on collision. A very detailed well written treatment of this restricted one-velocity theory is given. This theory is basic to the solution of more general and more realistic problems. Indeed, some of its results have immediate application in such problems as the slowing down of neutrons by elastic collisions and in other problems.

The book is organized into three parts: Part A, Introduction; Part B, Propagation in the absence of scattering collisions; and Part C, One-velocity theory of neutron differsion.

Part B discusses the solutions of the continuity equation for the angular demity function, $\psi(r, \Omega, t)$ the number of neutrons per unit volume and unit solid angle moving in the direction of the unit vector Ω; r and t denote the space coordinates and time respectively. The continuity equation is discussed for three types of streaming: with no sources in vacuum, with sources described by a function $q(r, \Omega)$, and with sources q and absorbers.

Included in this part are four place tables for the collision probability for a slab of half-thickness $\phi = a/2$, a sphere of radius a, and an infinite cylinder of radius a. These tables are listed in accordance with the values of b/l, a/l and a/l respectively where l is the mean free path. The variable a/l (b/l) ranges from 0.00 to 5.00 in steps of 0.01.

Part C starts with a discussion of the transport equation and some of its properties, discusses this equation for a uniform medium with isotropic scattering and finally shows how the results obtained for the uniform infinite medium may be applied to the solution of finite problems. In this part there are fairly complete tables and graphs of functions entering in the solution of the transport equation.

The Mathematical Tables Project is credited for the material in Tables 28a and 28b. The remaining computations reported in this work were carried out by the Los Alamos computing group under the direction of Bengt Carlson and Max Goldstein.

A. H. T.

127[S]—M. Ferentz & N. Rosenzweig, "Table of F Coefficients," U. S. Atomic Energy Commission Report ANL–5324, Argonne National Laboratory, Lemont, Illinois, 1955, 293 p. Price $6.30 (photostat), $3.00 (microfilm).

This report contains an extensive tabulation of a function $F_k(L, L', j', j)$ of five arguments known as the F coefficient, which occurs in the formulas for the angular correlation between successive nuclear radiations. The arguments of the function are restricted in accordance with various triangular and other conditions. The table is in two parts. Part I contains integral spins (7,189 entries on 151 pages) and Part II contains the half integral spins (6,413 entries on 133 pages). The values of F are given to eight decimal places. The computations were carried out on the UNIVAC at the AEC computing facility at New York University and the table was reproduced photographically from the original UNIVAC Output. The computations were checked by comparing the results with other computations and by computing $F_0(L, L', j', j)$ $(\equiv 1)$. On the basis of these checks it is asserted that the entries in the table are correctly given to the sixth decimal place.

A. H. T.

128[S].—A. Rahman, "Two centre integrals arising out of 2s and 2p atomic functions," Acad. r. de Belgique, *Cl. d. Sciences*, *Memoires*, v. 14, fasc. 2, publ. No. 1660, 1955, 13 p., 29 cm. Price 10 Belgian francs.

In working with the wave functions for diatomic molecules from 2s and 2p atomic wave functions centered on the atomic nuclei certain linear combinations of integrals previously tabulated by Kopineck [1, 2, 3] turn out to be very useful. These combinations of Kopineck's integrals are tabulated herein to five places.

A. H. T.

1. Hermann-Josef Kopineck, "Austausch- und andere Zweizentrenintegrale mit 2s- und 2p-Funktionen," *Z. Naturforschung*, B. 5a, 1950, p. 420–431.
2. Hermann-Josef Kopineck, "Zweizentrenintegrale mit 2s- und 2p-Funktionen II," *Z. Naturforschung*, B. 6a, 1951, p. 177–183.
3. Hermann-Josef Kopineck, "Zweizentrenwechselwirkungsintegrale III," *Z. Naturforschung*, B. 7a, 1952, p. 785–801.

129[S, V].—SVERRE PETTERSSEN, *Weather Analysis and Forecasting*, v. 1, *Motion and Motion Systems*, Second Edition, McGraw-Hill Book Co., Inc., New York, 1956, xix + 428 p., 23 cm. Price $8.50.

As the author states explicitly, this book is designed for use as a general text on weather forecasting. Accordingly, its subject matter and style of presentation are strongly slanted toward the interests and needs of the practicing meterologist, and will be of little direct value to the student of mathematics and numerical analysis.

Two chapters, however, deal with computing methods that are interesting, if only for their novelty. In chapter 3, the author outlines a method for extrapolating the position and intensity of certain definite features of atmospheric pressure patterns (e.g., "high," "lows," "fronts," etc.) from one moment to the next. This method is based entirely on the differential geometry of the pressure field and its instantaneous time-derivatives (assumed known). Thus, since it contains essentially no element of physics, it is equally applicable to other time-dependent fields.

Chapter 19 describes a graphical method, originally due to Fjortoft, for solving an equation of the form

$$\nabla^2\psi_t + \psi_x\nabla^2\psi_y - \psi_y\nabla^2\psi_x = 0$$

with ψ constant along a fixed closed curve in the (x, y) plane at all times t, and with ψ given everywhere within that curve at $t = 0$. Interpreted geometrically, this equation states that the instantaneous local velocity of the $\nabla^2\psi$-field is a vector equal in magnitude, but normal to $\nabla\psi$. This provides the basis for a graphical method of extrapolating the $\nabla^2\psi$-field at a slightly later time t, by moving the initial $\nabla^2\psi$-field for a short time t with the initial local velocity. The remaining problem, that of recovering the ψ-field at time t from the extrapolated $\nabla^2\psi$-field, is solved by a graphical method of repeated averaging—a procedure that is roughly equivalent to Richardson's method of relaxation. Although such graphical methods are doubtless very ingenious, it is the reviewer's opinion that they are a step backward against the current trend toward the more easily automatized digital type of computation.

Chapter 18, on Numerical Forecasting, was specially prepared for this volume by Dr. Arnt Eliassen of the Oslo University. This section of the book is a clear, concise, and remarkably complete account of the application of numerical methods and high-speed automatic computing techniques to the problem of weather prediction, and is probably of more general interest than the sections discussed above.

On the whole, Professor Petterssen's new book is clearly written, well illustrated, and represents a truly heroic effort to bring the practicing weather forecaster up to date with new and improved tools of his profession.

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130[S, V].—E. L. Harris & G. N. Patterson, *The Boltzmann H-Function Applied to the Shock Transition*, Institute of Aerophysics, University of Toronto, UTIA REPORT No. 40, 1956, iv + 17 p. + 8 p. of figures + 6 p. of tables, 28 cm. Available from Institute of Aerophysics on an exchange basis only.

The authors evaluate an approximation to the Boltzmann H-function for a gas with a shockwave present, where

$$H(x) = n \int f(\xi, x) \log nf d\xi,$$

n is the number of molecules per unit volume, and $nf(x, \xi)d\vec{x}d\vec{\xi}$ is the number of molecules with velocity components in the range ξ_i to $\xi_i + d\xi_i$ in the volume containing the point between x_i and $x_i + dx_i$ $(i = 1, 2, 3)$. The function $f(\xi, x)$ is approximated by using the solutions of the hydrodynamical equations of conservation of energy, means and momentum. Numerical solutions of the latter equations are involved in the calculations. The results of these calculations are listed for Mach numbers 1.5, 2.00, 2.3238, 3.00, 3.3764 and 4.00.

A. H. T.

131[X].—Cornelius Lanczos, *Applied Analysis*, Prentice Hall, Inc., Eaglewood Cliffs, N. J., 1956, xx + 539 p., 21 cm. Price $9.00.

This book is at first an abbreviated account of the author's work in different fields of numerical analysis. Therefore the reader may take it as a guide to the papers published by Lanczos in various periodicals. The specific contributions of Lanczos to numerical analysis can be listed as follows. First of all he discovered the usefulness of Chebyshev polynomials in many fields where they had not been used before. The reader will find much valuable material in this connection, in particular for solving large linear systems of equations by relaxation, for solving algebraic equations, for telescoping power-series and for the approximation of solutions of linear differential equations ("τ-method"). The author's method of computing eigenvalues of matrices by "minimized iterations" is outlined a little sketchily in chapter III and his more recent invention of "spectroscopic eigenvalue analysis" is described.

But the book gives moreover a general view of numerical analysis. It is very interesting for the specialist who will find even in its more elementary parts Lanczos' personal attitudes and his sometimes very original ideas. It may however be a little hard to read for the beginner. Chapter I deals with algebraic equations and much weight is given to Bernoulli's method called the "method of moments." Stability is discussed by mapping the left half-plane onto the unit circle. One should observe that by this procedure the nearest root to the imaginary axis does not correspond necessarily to the nearest root to the unit circle. Chapter II begins with a theoretical introduction in linear algebra and proceeds to the numerical methods of solving linear equations and eigenvalue problems. Iterative methods are given in Chapter III.

Chapter IV is devoted to harmonic analysis. One of the merits of the book is the extensive discussion of the numerical methods connected with network analysis and Laplace transforms. In particular the author develops four methods for

numerical inversion of Laplace transforms. The chapter is a little heavy because some results are discussed twice in the languages of Fourier and Laplace transforms.

The more classical problems of data analysis (interpolation and smoothing) and quadrature are treated in Chapters V and VI. The reader will find some new and interesting points of view, for instance the improvement of Simpson's rule by end corrections and a very elegant treatment of Hermite's quadrature formulas based on Legendre polynomials.

Error estimates are sometimes a little vague. As an example the estimate 6–5.4 for Simpson's rule can be made exact by multiplying the right side by 5/4 and assuming that the fourth derivative of the integrand does not vanish inside the interval of integration.

At the end of chapter VI there is an interesting application of Hermite's quadrature to eigenvalue problems connected with linear differential equations.

The last chapter of the book gives various methods for the expansion into power series and analytical extension.

It seems to the reviewer that the modern literature on numerical analysis has been taken into account up to 1952 approximately. Throughout the whole book almost every computational method is illustrated by numerical examples.

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132[X].—HEINZ RUTISHAUSER, *Der Quotienten-Differenzen-Algorithmus*, Birkhauser Verlag, Basel, Switzerland, 1957, 74 p., 23 cm. Price DM 8.50.

The Quotient-Difference Algorithm in its simplest form is a numerically effective method for determining the poles of a rational (or meromorphic) function from one of its Taylor expansions. Based on ideas due to Hadamard, Aitken, Lanczos, and Stiefel, and on concepts of the theory of continued fractions, the algorithm was developed by Rutishauser in three articles in *Z. angew. Math. Physik* [1, 2, 3] and shown to be an efficient tool for the solution of a number of fundamental problems of numerical analysis.

Each of the mentioned articles forms the backbone of one of the chapters of the work under review, but there are many changes and additions. Chapter I furnishes the theoretical basis of the algorithm. Among the new material we mention the didactically valuable rhombus rules (§3) and an addition theorem for continued fractions (§10). The latter appears to be a contribution to the formal theory of continued fractions and has several applications in the two later chapters. Chapter II deals with applications of the Quotient-Difference Algorithm to the summation of slowly converging series, to the factorization of polynomials, and to the interpolation of functions by sums of exponentials. Much of the new material included here tends to bring the algorithm closer to the computer. One of the disadvantages of the original form of the algorithm is that it occasionally may require dividing by small numbers. It is now shown (§9) how this can be circumvented by a simple device. Also, the quadratically convergent modification of the algorithm given in the original article for determining the real roots of a

real polynomial is now extended so as to yield also conjugate complex roots (§8). The original method for interpolation is now replaced by a numerically more stable process (§10). The section dealing with the summation of series is perhaps the least impressive of the chapter. No attempt is made to define a class of series for which application of the algorithm may speed up convergence. Chapter III describes the application of the algorithm to the determination of the eigenvalues and eigenvectors of a not necessarily symmetric or hermitian matrix. Again the author shows his concern with actual computation by including detailed flow diagrams and (as in the other chapters) numerical examples. In three appendices the author discusses several extensions of the algorithm. One of these, the so-called L-R-algorithm, is treated more fully by Rutishauser in an article to appear in volume 49 of the National Bureau of Standards Applied Mathematics Series.

On the whole the presentation follows the pattern of the underlying papers in *Z. angew. Math. Physik*. A few proofs that were originally missing (e.g., of Theorem 1 of chapter III) are now carried out in full. The algorithm is still based on the analytic theory of continued fractions, and the reader's task is not made easier by the fact that no references are made to Perron's standard works on the subject. [In an article which will appear also in the above-mentioned volume 49 of the NBS Applied Mathematics Series, the reviewer has given an introduction to the algorithm which is based solely on the elementary theory of complex variables.] The printing of the whole work is excellent, and the fraction bar has ample opportunity to demonstrate its much-neglected faculty to make complicated formulas easier to read.

The world of computation owes to the author already several pioneering contributions on automatic coding and on the numerical integration of ordinary differential equations. By his impressive work on the Quotient-Difference Algorithm and its ramifications, Rutishauser has firmly established his position as one of the most resourceful and inventive numerical analysts of the present.

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1. Heinz Rutishauser, "Der Quotienten-Differenzen-Algorithmus," *Z. angew. Math. Physik*, v. 5, 1954, p. 233–251.
2. Heinz Rutishauser, "Anwendungen des Quotienten-Differenzen-Algorithmus," *Z. angew. Math. Physik*, v. 5, 1954, p. 496–508.
3. Heinz Rutishauser, "Bestimmung der Eigenwerte und Eigenvektoren einer Matrix mit Hilfe des Quotienten-Differenzen-Algorithmus," *Z. angew. Math. Physik*, v. 6, 1955, p. 387–401.

133[X, P].—G. Temple & W. G. Bickley, *Rayleigh's Principle and its Applications to Engineering. The Theory and Practice of the Energy Method for the Approximate Determination of Critical Loads and Speeds*, Dover Publications, Inc., New York, 1956, vi + 157 p., paper bound. Price $1.50.

The authors state: "The object of this book is to explain and justify a rapid method of determining the fundamental period of a vibrating system or the condition of stability of an elastic system, with the degree of accuracy usually demanded in engineering problems. . . . The fundamental principle is due to the third Lord Rayleigh, and it applies not only to vibrating systems with a finite

number of degrees of freedom, but also to continuous systems such as a stretched string or metal reed."

The authors only mention that the Rayleigh Principle is intimately related to the principle of least action and state that whenever the differential equations of a problem, dynamical or otherwise, are equivalent to a variational principle, the problem is always soluble by methods analogous to those discussed in this book. However, this intimate relationship is nowhere discussed nor is the relation between variational principle for the system under discussion and the Induction Function $G(x, s)$ (the Green's function of the equations describing the system mentioned).

An outstanding feature of this book is the use of this function $G(x, s)$ to generate sequences of functions $f_n(x)$ which approach the first (and higher) proper function of the equations describing the system and the use of these approximate proper functions for the determination of approximations to the proper values of the system. The major portion of the discussion is devoted to a method for determining the first proper function and proper value. However lower and upper bounds are given for the latter quantity and the lower bound is shown to depend on the ratio of the first two proper values. A method for estimating this ratio is also given.

The introductory chapter gives a general account of Rayleigh's Principle, of the problems to which it can be applied, and of the results obtained in subsequent chapters.

Chapters I, II, V, and VI contain many illustrative examples in which the methods derived in some detail in Chapters III and IV are applied. Chapters I and II are mainly concerned with problems in elastic stability and Chapter V deals with vibrating systems.

Chapter VI deals briefly with theorems relating the fundamental frequencies of two systems which themselves are related by inequalities in their mass distributions, or potential energies.

Chapter VII entitled Numerical and Graphical methods discusses one problem: The determination of the fundamental frequency of a tapered Aeroplane strut.

Chapter VIII gives a few examples of the determination of equilibrium configurations of elastic systems.

The book is extremely well written and will be particularily useful to those who desire closed formulas for various approximations in engineering problems. Modern numerical analysts and other mathematicians will find this a thought-provoking book and very suggestive of methods for solving engineering problems. However it is an open question as to how suitable the methods discussed in this book are if numerical solutions are to be obtained by the use of automatic digital computers.

A. H. T.

134[X].—J. Greenstadt, "A method for finding roots of arbitrary matrices," *MTAC*, v. 9, 1955, p. 47–52.

The author proposes to triangularize an arbitrary matrix by a sequence of transformations of the form

$$A \to B = S^*AS$$

where S is a unitary matrix operating on two rows and columns and S^* denotes the transposed matrix. No proof is given that the sum of the squares of the absolute values of the sub-diagonal elements decreases to zero. It is known that this sum does not decrease monotonically. A variety of matrices whose elements were picked at random were treated by this method on an IBM 701 and it was shown that for this class of matrices the method converged.

A. H. T.

135[X].—E. Bodewig, *Matrix Calculus*, Interscience Publishers, Inc., New York, 1956, xi + 334 p., 23 cm. Price $7.50.

Experience shows that at least half the problems of a modern computing organization involve "linear algebra." Computers, in particular, will therefore welcome *Matrix Calculus*, which represents the first attempt to cover the whole field in a single volume.

The book is in four parts. Part I, in 85 pages, treats the theory needed in practical applications. It considers elementary properties of vectors and matrices; errors in A^{-1} caused by small uncertainties in A; measures of magnitude of A; decomposition theorems and orthogonalization; properties of latent roots and latent vectors; bounds for eigenvalues and determinants; and some theory of elementary divisors.

Part II, in 100 pages, considers practical methods for solving linear simultaneous algebraic equations. Direct methods include the elimination and condensation methods of Gauss and Jordan, both for exact and approximate solution; applications of matrix decomposition into triangles and its connections with elimination; the Gauss-Doolittle process, the methods of Banachiewicz and Cholesky, and Aitken's "triple-product" method. Details are given of desk-computing techniques and their variations of advantage for more automatic equipment. There is also a discussion of ill-conditioned equations and the "error" (accuracy) and "exactness" (precision) of solutions.

Iterative methods discussed include those of Gauss and Seidel, the acceleration devices of Aitken, and the steepest-descent methods of Hestenes, Stiefel, and others. The general theory of iteration, with theorems on convergence and its acceleration, is given at length. Methods for improving an approximate inverse, procedures which always converge and which are therefore considered suitable for automatic computing, and the practical advantages of various scaling operations, are also included.

The inversion of matrices, in about 50 pages, is the topic of Part III. In addition to the obvious applications and extensions of the methods of Part II there is a discussion of partitioning and bordering of matrices, an old but little known method for reducing a matrix to diagonal form, and a lengthy account of special treatment of the matrices arising in geodetic surveying.

The last 100 pages belong to Part IV, the determination of latent roots and vectors. Iterative methods are used to determine the dominant root and associated vector, and perhaps one or two sub-dominant solutions if convergence is slow, in this respect an advantage. For accelerating convergence, or causing convergence to a wanted solution, methods of "vector-deflation" and "matrix-deflation" are

fully explained. The construction of the characteristic equation is also discussed, together with its ill-conditioned dangers. There is a mention of Aitken's sequences for "higher" eigenvalues, and a further discussion of elementary divisors. Then come the orthogonal transformation methods of Jacobi and Givens, perturbation methods for improving accuracy, the gradient methods of Hestenes and Karush and the use of Rayleigh's quotient.

Some direct methods are also mentioned, associated with the names of Leverrier, Krylov, Duncan, Hessenburg, Samuelson, and others. These methods effectively form the characteristic equation, and find favor only for matrices of low order.

There is a bibliography of 111 references and a somewhat terse index.

Numerous examples illustrate the text (even in Part I), the amount of computation in each method is carefully assessed, and there are many computing hints, close attention to checks, and a willingness, refreshing if not always believed, to give an opinion on the relative merits of the different methods, according to the nature of the matrix, the amount of information required about it, and the computing equipment available.

Few omissions come readily to mind. Perhaps the "escalator" method was worthy of passing mention, however, and the absence of the name of Lanczos, from the section on latent roots, is somewhat surprising.

This reviewer did not find the book easy to read. The emphasis on a "new matrix calculus," with stress on rows and columns, avoidance of the use of individual elements, and accompanying new ideas and notations, together with the vast amount of material contained in relatively few pages, cannot readily be assimilated from the depths of an arm chair. But there is nothing essentially difficult, and the student who really makes the effort with this book and succeeds, can claim, with its author, to be an expert in this field.

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136[X, I].—H. Fishman, "Numerical integration constants," *MTAC*, v. 11, 1957, p. 1–9.

Tables of the b_j and x_j in the approximate quadrature formula

$$\int_0^1 x^n g(x)\,dx \doteq \sum_{j=1}^{m} b_j g(x_j)$$

are given for $n = 0(1)5$, $m = 1(1)8$ to 12D. They were prepared on an IBM 650 and are an extension of tables of P. C. Hammer, O. J. Marlowe, and A. H. Stroud [1] which covered the range $n = 1, 2, m = 1(1)5$ and $n = 3, m = 1(1)4$, but give 18D. The x_j are the zeros of certain orthogonal polynomials $q_m(x)$ defined by

$$q_m(x) = \sqrt{(n + 2m + 1)} \sum_{k=0}^{m} (-1)^k \binom{m + n + k}{m} \binom{m}{k} x^k$$

which are given explicitly for the same range of m, n. The b_j are defined by

$$b_j = \left\{ \sum_{i=0}^{m-1} [q_i(x_j)]^2 \right\}^{-1}.$$

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1. PRESTON C. HAMMER, O. J. MARLOWE, & A. H. STROUD, "Numerical integration over simplexes and cones," *MTAC*, v. 10, 1956, p. 130–137.

137[X, Z].—F. J. WEYL, *Report on a Survey of Training and Research in Applied Mathematics in the United States*, Monograph No. 1, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1956, vi + 58 p. Price $2.00.

Durch die Erfindung der Rechenautomaten und die schnelle Entwicklung der numerischen Analysis ist der vorher ruhig dahinfliessende Strom der mathematischen Forschung in den letzten Jahren etwas turbulent geworden und wir stehen heute vor manchen Problemen grundsätzlicher Natur hinsichtlich des Verhältnisses der Mathematik zu Naturwissenschaft und Technik und hinsichtlich des mathematischen Unterrichts.

Das dringende Bedürfnis nach Neufestlegung der Standorte und abklärender Aussprache hat in den vereinigten Staaten zum vorliegenden Bericht geführt, der auf Grund von zwei Konferenzen und einer allgemeinen Umfrage bei den Universitäten, der Industrie und den militärischen Organisationen von der mathematischen Abteilung des national research council ausgearbeitet wurde.

Die Aufgabe war erstens, Charakter und Umfang der gegenwärtigen Forschung in angewandte Mathematik festzustellen und nach pôtentiellen Möglichkeiten zur Vertiefung zu suchen und zweitens, Anregungen für die Ausbildung angewandter Mathematiker zu geben.

Der Bericht umgrenzt zunächst den Begriff "angewandte Mathematik," der ja sehr schwankend gebraucht wird, und erklärt sehr zutreffend, dass angewandte Mathematik weniger ein Sachgebiet kennzeichnet als eine geistige Haltung, charakterisiert durch die bewusste Bereitschaft zur Zusammenarbeit mit Schwesterwissenschaften, mit dem grossen Ziel, unsere Umgebung und uns selbst besser zu verstehen. Mit Recht wird darüber geklagt, dass viele der führenden Mathematiker, welche die Atmosphäre des Hochschulunterrichts bestimmen, nicht imstande sind, Substanz und Ziel dieser Denkweise zu begreifen.

Er stellt weiter fest, dass die nächste Zukunft durch eine niedagewesene Mathematisierung gekennzeichnet sein wird, die sich nicht nur auf Physik und Technik beschränkt, sondern auch alle unsere Lebensgewohnheiten beeinflussen wird. Als Gründe dafür werden angegeben einmal die wachsende Kraft mathematischer Methoden, die Notwendigkeit feinerer Berechnungen in allen Zweigen der Technik und ferner das Erscheinen der Automaten.

Es werden die vielen verschiedenen Wege beschrieben, die von den einzelnen Universitäten und Amtsstellen beschritten worden sind, um der steigenden Nachfrage nach angewandter Mathematik gerecht zu werden. Vor allem ist hier die neue Erscheinung des mathematischen Instituts zu nennen, dessen Erfolge

mehr auf der Aktivität einer Arbeitsgruppe beruhen, als auf spektakulären Leistungen Einzelner. Es wird auch durchaus hervorgehoben und als positiv gewertet, dass die Regierungsstellen und die militärischen Instanzen durch ihre Kontrakte oft diese Institute erst lebensfähig machen. Wiederum beklagt sich der Bericht darüber, dass viele Mathematiker diese nationalen Notwendigkeiten nicht einsehen und Kritik üben statt am Aufbau mitzuwirken. Es wird auch auf die drohende Gefahr hingewiesen, dass die mathematischen Wissenschaften in zwei Teile—den abstrakten und den angewandten—auseinanderfallen.

Das untersuchende Komitee findet aber, dass die Entwicklung zu langsam und zu unausgeglichen vor sich geht. Vor allem fehlt es an qualifizierten Studierenden, am Unterricht und an einer klaren Konzeption desselben. Die beunruhigenden Nachwuchsprobleme werden analysiert und auf verschiedene Quellen zurückgeführt.

Aehnliche Unzulänglichkeiten werden hinsichtlich der Rolle des Mathematikers in der Industrie festgestellt. Obwohl die Industrie in hohem Mass Mathematik konsumiert, stellt sie selten Mathematiker an, sondern übergibt mathematische Aufgaben ihren Ingenieuren.

Der Bericht gipfelt in folgenden Empfehlungen:

1) Es soll ein ständiges Komitee gebildet werden, das die Beziehungen zwischen der Mathematik und ihren Anwendungen fördert.
2) Die Universitäten sollen Forschungsprogramme in angewandter Mathematik für fortgeschrittene Studierende durchführen.
3) Alle Mathematikstudenten sollen gezwungen werden, sich auch mit anderen Wissensgebieten bekannt zu machen, in denen die Mathematik gebraucht wird.
4) Mathematiker aus der Industrie oder aus staatlichen Organisationen sollen zu Gastvorlesungen an Universitäten eingeladen werden.
5) Alle Dozenten der Mathematik sollen angehalten werden, Kurse in angewandter Richtung zu geben, sofern sie dies imstande sind.

Ein zweiter Teil des Berichtes befasst sich mit einigen typischen neueren mathematischen Entwicklungen. Es wird zunächst die axiomatische Methode geschildert, sodann die Rolle der Wahrscheinlichkeit und Statistik in den Theorien der mathematischen Physik auseinandergesetzt und die Bedeutung einer flexiblen und anpassungsfähigen Anwendung mathematischer Methoden in der Technik besprochen. Die ganz jungen Gebiete des "operational research," der mathematischen Theorien biologischer und soziologischer Phänomene werden gestreift und die numerische Analysis als Ganzes ist untersucht.

Als Anhang zum Bericht findet man eine Analyse der Antworten auf den vom Komitee versandten Fragebogen.

Es wurden hier absichtlich diejenigen Teile des Berichtes besonders gewürdigt, die unabhängig von den spezifisch amerikanischen Verhältnissen weltweite Gültigkeit haben. Alles Gesagte trifft wörtlich auch auf europäische Verhältnisse zu und zwar in verschärftem Mass, da in manchen europäischen Staaten auch noch das aktive Interesse der Behörden fehlt und somit die finanziellen Mittel oft knapp sind. Ferner veranlassen die besseren Arbeitsverhältnisse in Amerika sehr viele und oft die besten jungen Mathematiker aus Europa zur Auswanderung.

Der Wunsch nach einer gründlichen Neuorientierung des mathematischen Unterrichts hat einige europäische Länder veranlasst, internationale Besprechungen durchzuführen und die Fragen gemeinsam zu regeln. Eine erste Konferenz fand im Juni 1957 in München statt am mathematischen Institut der dortigen technischen Hochschule. Vertreter von Deutschland, Frankreich, Oesterreich und aus der Schweiz arbeiteten eine Resolution aus, die sich weitgehend mit den Forderungen des amerikanischen Berichtes deckt und in mehreren Fachzeitschriften (z.B. Zeitschrift für angewandte Mathematik und Physik, Heft VIII/5) veröffentlicht wird. Die an der Tagung gehaltenen Vorträge sollen ausserdem vom Münchner Institut herausgegeben werden.

Allen Fachleuten ist es wohl klar, dass angewandte Mathematik heute hauptsächlich ein Erziehungsproblem ist. Es handelt sich weniger darum, immer schnellere Automaten zu bauen, als das Personal heranzubilden, das imstande ist, mathematische Probleme schnell genug für den Automaten vorzubereiten.

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138[Z].—Milton H. Aronson, Editor, *The Computer Handbook*, First Edition, 55 p. *The Computer Handbook*, Second Edition, 1956, 71 p., The Instruments Publishing Co., Pittsburgh, Pennsylvania, 28 cm.

The first edition of this booklet contains six papers presented at the First Automation Exposition (New York, Nov. 29–Dec. 2, 1954) and published in the magazine *Instruments and Automation*. The second edition reprints three of these and adds eight others. In both cases about half of the papers are on analog computers and applications, the other half being on digital computers and applications.

The articles are written for the general reader, rather than being highly technical. They provide a limited amount of information about a number of specific computers and their business or industrial applications. The articles will not be useful to any person desiring detailed technical information about the computers discussed.

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139[Z].—*Proceedings of the Symposium* (Automatic Coding), 1957, Monograph No. 3, *J. of the Franklin Institute*, Philadelphia, Pa., 1957, vii + 118 p. Copies may be obtained from AUTO CODE, Franklin Institute, Philadelphia 3, Pa. Price $3.00.

The monograph contains the following papers:

AUTOMATIC CODING AT G. E. by Richard M. Petersen
SYSTEMS OF DEBUGGING AUTOMATIC CODING by Charles Katz
PRINT 1—AN AUTOMATIC CODING SYSTEM FOR THE IBM 705 by Robert W. Bemer

THE PROCEDURE TRANSLATOR—A SYSTEM OF AUTOMATIC PROGRAMMING by Henry Kinzler and Perry M. Moskowitz

OMNICODE—A COMMON LANGUAGE PROGRAMMING SYSTEM by Russell C. McGee

A MATRIX COMPILER FOR UNIVAC by Laurence C. McGinn

A MATHEMATICAL LANGUAGE COMPILER by Alan J. Perlis and Joseph W. Smith

A MECHANIZED APPROACH TO AUTOMATIC CODING by E. C. Yowell

There is a short introduction by John S. Burlew of the Franklin Institute, a transcription of the discussion of each paper, and a list of those attending the Symposium.

It seems to be a sad fact that workers in the field of automatic coding, who in general are scientific or research personnel, bring a most unscientific and unresearchlike attitude to these symposia. The papers presented at this symposium are, with two and possibly three exceptions, simply descriptions of individual and special-purpose coding systems which are of interest to users of the particular computer for which they were written, and to nobody else. Most of the information contained in these papers could more easily and efficiently have been obtained by interested parties through a users' organization; by this time such organizations have grown up about most large computers. Evidence of the inefficiency of communication is the turn generally taken in the discussions; questions are often requests for information already given in the paper, and comments generally consist of destructive or petty and obvious criticism. Constructive criticism—the contribution or generation of constructive ideas—is so rare as to be negligible.

Some of the papers deserve at least individual mention; one such is the Katz paper. This contains a short discussion of debugging problems that arise in automatically coded programs, which are difficult of solution because, by the very nature of automatic coding, the programmer is insulated from (and may be completely unfamiliar with) the computer. Mr. Katz describes an attempt to produce debugging routines for some compilers for UNIVAC. Automatic debugging is a most interesting and difficult logical problem, and this paper is one of the few in which more rather than less detail would have been welcome.

The Perlis-Smith paper is another refreshing exception. It is, in the final analysis, a description of an automatic coding scheme for the IBM 650, but the language and organization are a pleasure to see; further, the approach to the problem is at once scientific and comprehensible. There is a clear, concise description in logical terms of just what the automatic coding problem is, and a clear and convincing justification of the methods used in solution. It may or may not be significant that this is the only paper arising from work at a university.

The last paper, and the last deserving mention, is the Yowell paper. This is quite different from the others in that it describes an attempt (by The National Cash Register Company) to design a computer (the 304) with macro-commands; this is automatic programming by hardware rather than routines. The paper is at once wordy and short, but the novelty of the content makes it interesting.

The tenor of papers and comments makes it clear that the overwhelming majority of those attending the symposium have a distorted idea of just who and what automatic coding is for. Time and again the emphasis is on simplifying programming so that cheaper talent can be used to write programs. This is a case of being right for the wrong reasons. The saving in time and money in making do with low-priced programmers is as nothing when compared with the saving that can be effected if high-priced talent—scientists, engineers, executives—can have access to a computer without having to explain problems or learn programming. The only statement or comment made at the Symposium in support of this philosophy comes from Dr. Perlis and is, in part: "It is the man who proposes the problem who is important, and his ideas are the ones that count."

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140[Z].—D. D. MCCRACKEN, *Digital Computer Programming*, John Wiley and Sons, Inc., New York, 1957, vii + 253 p., 24 cm. Price $6.50.

Digital Computer Programming offers for the first time an excellent introduction to the problems of coding for a high speed stored program computer. Such a work has long been needed in this field; McCracken's book will probably remain as the standard of comparison for some time. Since this book will undoubtedly be widely used, not only as a textbook in colleges but also as a training manual within industry, it is felt that a fairly detailed review of the work is indicated. In what follows, the plan of attack will be to cover the book more or less chapter by chapter, and, in general, point up some of its shortcomings. Most of these arise from viewing the book from the point of view of a neophyte in coding, and will possibly seem obvious to the old hands in the business, who can read *Digital Computer Programming* almost as if it were a novel, exclaiming "Yes, yes!" at every page. Following this survey will come a few remarks on the general philosophy of the book.

In chapter 1 the newcomer is given a brief introduction to stored program computers. It becomes obvious almost immediately that the work is intended as a textbook, as new terms are introduced with fair rapidity. Most of these terms are indicated in italics, and defined contextually, which makes for easy reading, but these terms must be mastered before going on, and will probably require several readings on the part of the student. In order to have a concrete machine with which to work, McCracken invents TYDAC, the TYpical Digital Automatic Computer, a decimalized version of a modern binary scientific machine. A brief history of computing follows, which is fairly complete with one exception. Most of the major users of large scale machines for scientific computing started out by applying standard punched-card machine techniques to engineering calculations, and many of the methods, traditions, and customs which the newcomer to scientific computing encounters have their roots in this early work. Some credit should be paid to these pioneers.

A fairly comprehensive outline of the steps in coding skips mention of one of the most important steps—that of making sure the proposed problem is really

the one which should be solved. Often a great deal of labor may be saved by incorporating several small problems into one large systems analysis. On the other hand, it may be too early in the investigation of a problem to tackle more than a small piece at a time. The large amount of time spent recoding problems points up a need for more time spent in this phase of solution. Also, no mention is made of the large amount of feedback that ordinarily occurs between the scope of the problem under solution and the method employed.

There is a good survey here of typical applications of digital computers; this might also have included some examples of problems which are *not* suited to high-speed stored program machines.

A lucid, though conventional, explanation of coding fundamentals is presented in chapter 2. The going is still fairly fast here, and much reference will be made back to the terms previously defined in chapter 1, and to the appendix which contains a complete summary of TYDAC. (A worthy addition to the book would have been a small card, loosely inserted, containing a summary of the instruction codes, etc. Practically every coder for a real machine carries one of these supplied by the manufacturer; TYDAC should not be an exception.) Cogent points which must be remembered are often hidden in the middle of a paragraph; the book will probably be much underlined.

A good glossary of computing terms would be welcome here. Since this text will find such widespread usage, it is to be regretted that the author did not include what might have become a definitive dictionary of computing terms.

There is a good description of the execution of a three step program which adds two numbers, but the section on multiplication and division processes within the machine is probably overdone. A better use of space here would have been to present the arithmetic registers as they appear before and after the execution of the instructions, as is done in the section on scaling. It is pleasant to see an introductory work which mentions the necessity of clearing the MQ register before division takes place, although more emphasis might have been placed here—especially in connection with small-integer arithmetic.

In an example on p. 26 several opportunities have been missed to point up the possibility of subtle errors in even such a simple problem as $(10 \cdot A + B)C$. Here, as happens throughout the text, the examples seem fairly simple when the correct answer is given. Only three examples of coding in the entire text are pointed out as containing errors—certainly a far smaller proportion than maintains in real-life coding. A very worthwhile addition to the explanations would give examples which do not work, and the reasons why.

The exercises appended here, as throughout the work, are excellent. Perhaps the most frequently neglected part of teaching coding is in this realm, and it is refreshing to see examples more typical than $A + B = C$.

Chapter 3 leads us painlessly through the mysteries of binary and octal number systems. One addition to this chapter that could be made would explain the use of mixed-base number systems for compacting information within a word, as is often done to minimize input-output time or conserve storage space.

The Decimal Point Location Systems, as described in chapter 4, are exceedingly well done. McCracken is to be congratulated on his fearless approach to the so-called "mysteries" of fixed-point coding. There are many experienced coders

who could profitably study this explanation. The first method, that of fixing the point in the middle of the word, seems rather awkward and artificial, and should probably have been cut a bit. Here would be a good place to explain the necessity of clearing the MQ before division, but this is not done.

The "graphic" method has long been the favorite of this reviewer, who finds here the first really good explanation of the method. The examples of multiplication miss the fact that in a large proportion of cases a left shift is necessary after the multiply, since both factors may not reach their maximum value at the same time. Also neglected is the problem of finding a full ten digit rounded quotient, which is never presented in the book. The scaling of the remainder after a division is never presented; this should be included. Some simplification could have been made by considering only the number of digits before the point, as the number after is redundant. To be complete, this method should include a set of rules similar to the 5 step rule given under the scale factor method, and would probably give the "graphic" method more widespread use.

The commonly used scale factor method is accompanied by a good set of rules for its employment, and should dispel much fear on the part of the coder approaching fixed point problems.

In the section on scaling, the author has missed an opportunity to present the coder with a check on the programming of the problem that is often valuable. In coding fixed point arithmetic, the values of the shifts required should normally be small; e.g., 3 or 4 places at the most. If a shift of eight or nine is arrived at in, say, an addition, an insignificant quantity is probably being added. The statement of the problem may be checked at this point, and many errors have been detected in such a fashion.

The coder should be warned, in conjunction with the problem of overflow, that in a large majority of cases the overflow indicator will be turned on by some part of the problem which does logical work, or by subroutines, and that it should be turned off prior to embarking on a sequence of fixed point arithmetical calculations and checked immediately following the sequence.

In this section, as elsewhere, a good deal more space could have been profitably devoted to pointing out possible pitfalls. The examples are generally good, although more requiring "tight" scaling to retain significance could have been included.

Chapters 5 and 6, which deal with address computation and loops, lead the coder quite easily into the real power of stored program machines. Particularly to be commended is the decision to separate the address computation from the testing computation; introductory examples here are often much too complicated. Two objections were noted here. In several examples some of the necessary information is buried in the text, and it would have been better to write out cells containing instructional constants, etc., following the examples. Where this is not done, the somewhat confusing notation "Loc 1400" is used rather than the more common "L(1400)" or "Loc. of 1400."

This section would be a good place to point up the advantages of standardization when writing loops. There are many ways to write a given loop; the important point is for the coder to pick a single method to use and stick with it—the temptation to be clever is seldom resisted by the neophyte, and usually leads to disaster.

The excellent coder usually develops his own style of coding, and this becomes a major factor in reducing his own mistakes.

There is an excellent paragraph at the bottom of page 77 which bears repeating here.

> "Depending on the nature of the loop and of the test, it is possible to make a truly remarkable variety of mistakes in testing. If the loop should be carried out n times, it is quite easy to make mistakes which will result in doing it: (1) not at all; (2) $n - 1$ times; (3) $n + 1$ times; (4) $2n$ times; (5) until the power fails or the machine breaks down. It is fairly safe to say that loops, although one of our most powerful tools, are also a *very* large source of errors. Whatever other prechecking systems may be used, it is always advisable to go back and check the loop-testing parts of the program. As indicated above, one simple way is to analyze what would happen if the loop were to be executed only once."

This succinctly conveys to the coder the main source of errors in coding. McCracken might have pointed out that once a coder learns to "count to one" (not only in writing loops, but in many other phases of a problem) he is well on the way towards eliminating mistakes in coding.

On page 80 there is a discussion of the time vs. space problem which always seems to bother the new coder. Perhaps this would be a good time to inject the point that very often the best loop is not the shortest, not the fastest, but the one which works the first time. The amount of time spent in debugging a slightly faster or shorter loop often far overshadows the time saved in running the problem. Many coders adopt the policy of writing all loops in a standard fashion, checking the program out, and then going back and "cleaning up the code," one loop at a time, when time considerations are really important.

There is also a gem of wisdom at the beginning of chapter 6 which might have been emphasized more—in coding a loop, start from the inside and work out. This philosophy can be carried to all parts of a problem, and bears repeating many times, although the student may quickly find this out for himself in working out the fine examples at the end of this section.

The chapter on flow charting is well introduced and presented; unfortunately no attempt has been made to give general rules for problem organization. Moreover, no symbolism has been adopted for indicating the execution of a closed subroutine, whose detailed flow chart may be given elsewhere. Some attention to this point might have helped to keep flow charts from growing almost beyond comprehension.

TYDAC contains, as do most modern scientific computers, a set of index registers. A rather arbitrary, though adequate, set of instructions is provided for operating on them, and an extremely good explanation is made of the distinction between indexable and non-indexable instructions. Another example of a program with an error occurs here, and again the opportunity to point up the advantages of standardized coding is missed.

In chapter 9 the student is introduced to one of the most powerful tools at his disposal—subroutines. The various methods for entering and leaving a subroutine are well presented, but two methods for communicating information to the subroutine that are commonly used are not emphasized. (These are transmittal via

the machine's arithmetic registers and via standard cells in memory.) Several other points are missed that are vital to the discussion. A generally used subroutine must somehow take care of all possibilities of error in inputs, with appropriate error warnings, and must also conform to some prescribed set of conventions regarding the use of overflow indicators, index registers, etc. An example of a subroutine with an error exit is given in one of the examples in a later chapter, but the point is not emphasized. It is in the attention to details such as these that a good subroutine is distinguished from a poor one.

The statement is made, on p. 114, that it is not clear whether or not the use of index registers for linking to subroutines will grow. On the contrary, the reviewer feels that one of the most important advantages of the newer machines lies in their built-in features, such as the set index jump instruction and indirect addressing features, which permit subroutines to be constructed with great ease while coding. The closed nature of these subroutines greatly facilitates the check-out phase, and they will undoubtedly become a more and more important part of programming. It is at this point in the book that the coder might have been introduced to good coding practices, and it is unfortunate that the point was not made.

The introduction to floating point operation, as given in chapter 10, is generally good. His warning against the injudicious use of floating point is not too strongly made, in the face of the general trend toward hardware incorporating this feature. The new coder may be lulled into a false sense of security if he is not strongly confronted with the pitfalls that can arise. One of the most time-consuming problems for a programmer using built-in floating point operations in a large problem concerns the treatment of underflows and overflows; the routines necessary are not mentioned at all. It should also have been explained that built-in floating point operations are not always exactly analogous to the corresponding fixed point operations; e.g., TYDAC's MQ is cleared in floating point addition. This is often a big source of errors for the new coder.

Input-output methods can be treated only briefly in a book such as this, as McCracken points out, since these are strongly dependent on the actual machine used. This is unfortunate, as perhaps the best way for a coder to learn the differences between numbers, addresses, locations, operations, etc., and at the same time meet non-numeric and numeric data and conversion and scaling problems is to tackle an input or output routine. This is especially true on a binary machine. The statement that a coder need never know input-output will be contested by many, including this reviewer. The inclusion of a console on TYDAC leaves something to be desired in view of the general trend away from having the programmer touch the console, or indeed, operate the machine at all. There might have been a section included, more pertinently, on the problems of organizing the production phase of a problem so that it may easily be run by a professional operator who may have no knowledge at all of the coding or the problem itself.

A load program is presented which uses an initial address and a word count on each card to control loading and the advantages are detailed. The disadvantages should have been presented, too. Suppose the deck has had corrections added, and is somehow out of order? There is no indication that this has happened, and the program might run through to completion, giving wrong answers, at a large cost in machine time. Similarly, no indication is given if a card is missing; much

running time can elapse before this is detected. Many problems arise in connection with input-output procedures, and the programmer should be made aware of these early in his career.

The presentation of magnetic tape procedures is again necessarily brief. The inclusion of check-summing as a machine checking feature is good, especially in view of the fact that TYDAC does have a redundancy check indication. The necessity of having programmed checks on the machine operation, even when checking circuits are built in, could probably have been emphasized elsewhere with equal advantage. Here again the coder should be warned that his problem will be run by professional operators unfamiliar with his routine and that he should try to simplify tape handling procedures as much as possible.

Chapter 13 presents a basic introduction to program checkout techniques. Since the average coder spends the largest amount of time in this phase of problem preparation, a much more extensive treatment of the subject would be indicated. Unfortunately, no general set of rules to guide the programmer has as yet been presented. McCracken fails to distinguish the two phases of program checkout: debugging and testing. By debugging is meant the process of making sure that the program does what the coder meant it to do, and by testing is meant the process of making sure that the program solves the problem it is intended to solve. The distinction is strong, and the testing phase is often woefully neglected, even by experienced programmers.

The debugging process is fairly well covered, including the standard techniques of tracing, memory dumping, and breakpoint printing. Also welcome is a description of the "decoding" process of checking a code before it is tried out on the machine, and various methods are presented. An excellent example points up the necessity for picking realistic and non-degenerate test case data, and should decrease the number of times the cry, "But the test case checked perfectly!" is heard.

No rule is given for the procedure to be followed when the newly coded program collapses; a simple one is to try to get the program to execute all of the instructions from beginning to end without regard to the correctness of answers, and then to tackle the arithmetic portion of the code. All such rules are, of course, rather dependent upon the programming system used, and will vary from machine to machine. Another fruitful area for making mistakes is ignored—the procedure for correcting mistakes once they are found. Nowhere is this problem covered, and as the neophyte programmer does not have symbolic techniques at his disposal at this point in the text, he may reach the conclusion that it is necessary to do a great deal of rewriting each time an error is discovered. A few words as to the necessity for leaving room for patching would be in order here. Also, were symbolic methods available to the coder at this point, the ease with which these permit the inclusion of personalized diagnostics could be presented.

It is difficult, in a short chapter, to present relative (or symbolic) programming techniques and give a good indication of their power. The advantages of these techniques are most strongly felt on large problems, and a student is not inclined to go through a large problem example in a new and unfamiliar notation. It is significant, however, that once a coder has mastered relative and symbolic coding systems he usually becomes much more skilled in programming actual machine

language. The second exercise in this chapter requires the student to find a "small" error in a routine which has been presented in the text, and points up the fact that even advanced methods lead not to perfect programs, but to different types of errors, with which the coder must learn to cope.

The final sections on interpretive routines, double precision routines, and miscellaneous programming techniques are well written and the examples which accompany these chapters provide the opportunity for the student to apply the principles learned in the early part of the book.

In spite of the general level of excellence of this text, there are several areas which have been rather neglected if programming, rather than coding, is to be taught. The first of these is that of program checkout and testing, and the second is that of problem organization. In the majority of classes using this as a text, it is difficult to see how the student can be made aware of the many mistakes he will undoubtedly make in coding the examples given. Certainly few instructors will have time to discover, by the process of decoding, all of the subtle mistakes which an inexperienced coder can make in even the simplest examples. One solution which will undoubtedly be adopted at a few schools, is to code an interpretive routine for a high speed computer to simulate the features of TYDAC, and to provide laboratory sessions in which to try out the problems.

If this is not practical, or if a high speed machine is not available, the only alternative would seem to be to standardize coding techniques and penalize the student who uses fancy routines much as he would be penalized in actually trying to check them out on an operating machine. Certainly, as has been indicated in individual cases above, there are many opportunities in the text to point out the various mistakes that can be made, and it would be a great help if a greater portion of the text, and particularly the examples, was devoted to finding errors in routines. There is a trend, particularly among the users of the largest scientific machines, towards finding mistakes without the use of elaborate machine debugging programs. Examples might be given which would contain the original coding for a routine, and the results of running this coding on TYDAC. (This routine stopped at xxxx. Why?) If necessary, the relevant portions of a memory or console dump could be supplied.

Perhaps the greatest difference between the performance of a mediocre coder and a top-notch programmer lies in the way in which they organize a problem. It is a far cry from writing a set of twenty or fifty instructions to completing a problem containing many thousand orders, and many coders fail to bridge this gap. The good programmer thinks at all times about the possibility of errors, and so arranges his problem to eliminate as many mistakes before they occur as he can. One technique that is widely used is to break the problem into small, self-contained subroutines, trying at all times to isolate the various sections of coding as much as possible. At first glance this appears extremely wasteful of instructions—many of them are communicating information between routines that could well communicate directly with one another. It is in this direct communication, however, that many mistakes are made, and by isolating the various portions of the code as much as possible the problem is reduced to many much smaller ones. The truth of this seems very obvious to experienced coders, yet it is hard to put across to the newcomer. An extremely valuable addition to the McCracken text would be a rather

large problem, perhaps requiring as many as several thousand instructions, broken down into small logical blocks, with descriptions of these blocks. Similar problems could be presented for analysis by the student. This sort of work also has the advantage of being quite easily checkable by the instructor.

The experienced programmer, on reading the text, will probably wonder, as did the reviewer, why symbolic coding was not introduced much earlier rather than being relegated to a few pages at the end of the book. Certainly everyone who is introduced to computer programming in industry is given symbolic coding shortly after he learns the basic operation of the machine. The aim of symbolic coding is to help alleviate some of the burdens of coding, help him in checking and testing his program, and it is probably most useful to start the would-be coder off with as much help as possible, leaving him to concentrate on the essential problems of programming. Many excellent programmers would be lost without such an aid, and the student is probably no exception.

One final criticism of the book will be raised by many. Why, it will be asked, bother to teach many students to program for a mythical machine which they will never actually use, and bears only a confusing relationship to the machine they will ultimately program, and, additionally, one with so many instructions? This reviewer feels, on the contrary, that it is probably worthwhile for the coder to be familiar with more than one machine, even if one of them is mythical. Machines are continually being manufactured which are larger, faster, and with a larger set of instructions, and the switch from TYDAC to an actual machine will adequately prepare the coder for future machine changes.

In conclusion, it should be stated once more that *Digital Computer Programming* will probably be the standard text on coding for some time to come, and rightfully so.

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141[Z].—W. Sluckin, *Mind and Machines*, Penguin Books, Inc., Baltimore, Maryland, 1954, 221 p., 18 × 11 cm. Price $0.50.

This book attempts a comparison of human brains with digital and analogue computing machines in the endeavor to answer some questions raised by the existence of these machines. These questions: of similarity of mind and machine, of how the human brain works, of what constitutes "thinking," of whether machines "think," are all open fields for controversy. The author presents a many-sided discussion, leaving it to the reader to make his own decisions, and without always disclosing where he himself stands on some of these problems.

Cybernetics, and the concept of negative feedback are discussed in an attempt to determine how the human brain functions. Machines, particularly electronic machines, use the negative feedback principle extensively. How valid are our attempts to introduce this concept into our discussions of the nervous system? The author shows clearly the value and the limitations of the principle of negative feedback in this application.

In one sense, however, he persistently ignores a fact which to this reviewer should be more often used in these arguments, particularly on the question of

whether what a computing machine does constitutes real thinking. This fact is that man has built the machine and has put into it, by virtue of his designs, a predictable pattern of behavior. True, this may be a very complicated pattern, it may involve decisions simulating human thought processes, it may even simulate "learning." But in every case, in the final analysis, a machine has been built which can only do what the designer says it can do.

Thus it seems a little odd to take this simulator, this computing machine, and use it as a model of what we suppose the brain to be like. It would be very surprising if it did not resemble the brain in some sense, since we have built it with our own logical thought processes as a model. This seems to be a weakness of the arguments for a mechanistic interpretation of the human brain.

One minor criticism concerning usage should be made. The author states, on page 176, for example, that digital and analogue computing machines "think" in that they perform *mathematical* calculations. This is not correct in a strict sense. These machines perform *arithmetical* calculations.

This is an informative book, and while it avoids dogmatic answers to the problems it states, it does give a wide basis for consideration of validity of the many hypotheses which have been suggested.

W. Sluckin combines interests in two fields ideally situated to help in the discussion of these problems: he is both a psychologist and an electrical engineer.

Each chapter concludes with a pertinent bibliography for additional reading.

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142[Z].—*Elektronische Rechnenmaschinen und Informationsverabeitung* (*Electronic Computers and Information Processing*), Nachrichtentechnische Fachberichte (Technical Information Report), Friedr. Vieweg and Sohn, Braunschweig, Germany, v. 4, 1956, viii + 299 p., 29.5 cm. Price DM 26.

This volume reports the proceedings of an international meeting held in Darmstadt, Germany, from October 25th to 29th, 1955 under the sponsorship of the German Association for Applied Mathematics and Mechanics (GAMM), the German Society of Electrical Engineers (VDE), the German Mathematical Society (DMV), and the Association of German Physical Societies (VDPG).

Among the seven introductory papers, three were presented by U. S. representatives with H. H. Goldstine evaluating computer speeds required to solve significant differential and integral equations by existing methods; A. S. Householder discussing numerical mathematics; and H. H. Aiken projecting the future of automatic computing machinery. A. D. Booth of England discussed input-output systems. The problems of data processing, programming and switching and memory techniques were respectively considered by R. Piloty of the Technische Hochschule München, H. Rutishauser of the Technische Hochschule, Zürich, and H. Billing of the Max Planck Institute, Göttingen.

The largest group of papers concerned themselves with descriptions of European computer developments and will have value as source material not readily

found elsewhere. The computers reported on are tabulated below:

COUNTRY	*ORGANIZATION*	*MACHINE*
Austria	Institute for Low Frequency Techniques, Technische Hochschule, Vienna	Logical Function Computer
Belgium	Institute for the Encouragement of Scientific Research in Industry and Agriculture and National Fund for Scientific Research, Antwerp	IRSIA-FNRS
Czechoslovakia	ARITMA, Prague	Calculating Punch
	Institute of Mathematical Machines, Academy of Sciences, Prague	SAPO
		Digital Plotter
		Linear Equation Solver
		Relay Network Synthesizer
Germany	Institute of Applied Math. Tech. Hochschule, Dresden	D 1
	Institute of Electrical Information Handling Tech. Hochschule, Munich	PERM
	Institute for Experimental Mathematics, Tech. Hochschule, Darmstadt	DERA
	Leitz Optical Works, Wetzlar	Z 5
	Max Planck Institute, Göttingen	G 1
		G 2
		G 3
Netherlands	PTT Laboratories, Hague	PTERA
	Mathematische Centrum Amsterdam	ARRA
		ARMAC
Sweden	Matematikmaskinnämnden Stockholm	BESK
Switzerland	Institute for Applied Math. Tech. Hochschule, Zurich	ERMETH
U. S. A.	IBM	705
USSR	Academy of Sciences, Moscow	BESM
		URAL

Eleven papers describe approaches to programming in Germany and the Netherlands, with consideration given both to changes in machine design to facilitate programming and to compiling and interpretive methods.

In the area of numerical analysis a dozen papers cover problems in linear algebra, iterative processes, interpolation, hyperbolic partial differential equations,

quadratures, nonlinear differential equations, hydrodynamical equations, linear programming, aircraft design computation, weather forecasting, matrix inversion, and astronomy.

The concluding miscellaneous section includes a survey of logical function computers, a description of contact grids as an aid to relay network synthesis, a discussion of statistical programs in industry and description of an equivalence algebra for representation of digital machines.

Aside from the content of the papers, interested researchers will find a number of new references associated with several of the articles.

Moreover, the publisher has been quite considerate in providing a list of addresses of the authors as well as short translated summaries of all the papers.

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TABLE ERRATA

Reviews and papers in this issue mention errata in the following works:

A. J. C. Cunningham & H. J. Woodall, *Factorization of* $y^n \pm 1$, $y = 2, 3, 5, 6, 7, 10, 11, 12$ *up to high powers* (n), Francis Hodgson, London, 1925. (See Raphael M. Robinson note, "Some factorizations of numbers of the form $2^n \pm 1$," p. 265.)

R. B. Dingle, D. Arndt, & S. K. Roy, "The integrals

$$A_p(x) = (p!)^{-1} \int_0^\infty \epsilon^p(\epsilon + x)^{-1}e^{-\epsilon}d\epsilon \text{ and } B_p(x) = (p!)^{-1} \int_0^\infty \epsilon^p(\epsilon + x)^{-2}e^{-\epsilon}d\epsilon$$

and their tabulation," Review 119, p. 279.

D. N. Lehmer, *List of Prime Numbers from* 1 *to* 10006721, Review **107**, p. 272.

257. —C. A. Coulson & W. E. Duncanson, "Some new values for the exponential integral," *Phil. Mag.*, v. 33, 1942, p. 754–761.

A comparison of tables of $\mathrm{Ei}(x)$ and $-\mathrm{Ei}(-x)$ appearing in this paper with more elaborate tables by Harris [1] reveals a total of nine rounding errors and two more serious errors in the former.

Rounding errors appear in the 10-figure values given by Coulson and Duncanson for $\mathrm{Ei}(x)$ when $x = 20, 31, 46$, and 47, and for $-\mathrm{Ei}(-x)$ when $x = 16, 20, 23, 29$, and 39.

Their claim of accuracy to within 2 units in the last figure is refuted, however, by the following emendations. Corresponding to $x = 44$ and 45 their approximations to $\mathrm{Ei}(x)$ should be increased by nearly 8 units in the last place; that is, for 2.990444711, read 2.990444719, and for 7.943916028, read 7.943916036, respectively.

The accuracy of Harris's table of the interpolation coefficients $R_n(1)$ was confirmed by me by an independent calculation, and these data were then used to

verify the consistency of his 18-figure values for Ei(44) and Ei(45) with the adjacent entries in his table.

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1. FRANK E. HARRIS, "Tables of the exponential integral Ei(x)," *MTAC*, v. 11, 1957, p. 9–16.

258.—CARL-ERIK FRÖBERG, *Hexadecimal Conversion Tables*, C. W. K. Gleerup, Lund, Sweden, 1957, (*MTAC*, Review **82**, v. 11, 1957, p. 208.)

The following erratum has been found:

Page 11, *for* $0.^2$ 65 00D41 DF3B6 45A1D
read $0.^2$ 65 00D4F DF3B6 45A1D.

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NOTES

Acknowledgments to Referees

The editors of *Mathematical Tables and Other Aids to Computation* were happy to take advantage of the action of the Council of the American Mathematical Society at their December 1956 meeting in Rochester, New York, and furnish to them a list of persons who have refereed papers for *MTAC* during 1955–1956. This list was incorporated with names furnished by various other journals and published on pages 27 and 28 of AMERICAN MATHEMATICAL SOCIETY NOTICES, April 1957. The editors of *MTAC* deeply appreciate the services rendered by its referees.

C. B. T.

The Illinois Journal of Mathematics

The Illinois Journal of Mathematics is devoted to the publication of original research papers in pure and applied mathematics. The first two issues of this journal have appeared with the tables of contents listed below.

Contents, Vol. 1, No. 1 *March, 1957*

Contents, Vol. 1, No. 2 *June, 1957*

Manuscripts intended for publication in this journal may be written in English, French, German, or Italian and should be addressed to the Illinois Journal of Mathematics or one of the following editors:

Reinhold Baer, Mathematisches Seminar der Universität, Frankfurt am Main, Schumannstrasse 58, Germany.

J. L. Doob, Department of Mathematics, University of Illinois, Urbana, Illinois.

A. H. Taub, Department of Mathematics, University of Illinois, Urbana, Illinois.

George W. Whitehead, Massachusetts Institute of Technology, Cambridge 39, Massachusetts.

Oscar Zariski, Department of Mathematics, Harvard University, 2 Divinity Avenue, Cambridge 38, Massachusetts.

A. H. TAUB

Policy Committee for Mathematics, Technical Advisory Committee to the National Bureau of Standards, Annual Report, May, 1957

1. The Bureau of Standards Technical Advisory Committee for Mathematics has reviewed the work of the Applied Mathematics Division of the Bureau (Division 11) and finds that in spite of the difficulty this division is having in maintaining its present staff and acquiring new staff, it is providing very excellent mathematical, computational, and statistical services to other agencies of the government as well as to other Bureau divisions. The four sections of Division 11; Numerical Analysis Section (11.01), the Computation Laboratory (11.02), the Statistical Engineering Laboratory (11.03) and the Mathematical Physics Section (11.04), have interesting, useful and imaginative research and training programs which make them effective organizations for the discharge of their responsibilities.

2. The Numerical Analysis section (11.01) has continued its outstanding work in modern Numerical Analysis and the use of computing machines in pursuing research problems in mathematics. In addition it has inaugurated a training program in numerical analysis for senior university staff.

The purpose of this program is to give regular university staff heretofore specializing in fields different from numerical analysis, a training in that field which will enable them to direct the operation of a university computing center, and to organize training and research in numerical analysis on their return to their own institutions. Thirteen people began work on this program on February 11, 1957 and continued to be in residence at the Bureau of Standards until June 7, 1957. The funds for stipends to the participants and other expenses in connection with this training program were supplied by the National Science Foundation.

Trainees were taught programming for SEAC and were given a very intensive course of lectures on classical numerical analysis, linear equations and matrix inversion, quadrature and ordinary differential equations, characteristic values of matrices, partial differential equations, integral equations, linear programming, and non-linear equations. In addition lectures were given covering a variety of other topics including recursive functions and Turing machines, statistical calculations, discrete variable problems, and some experimental mathematical calculations.

The lecturers included, in addition to members of the staff of the Applied Mathematics Division of the Bureau, mathematicians of note presently engaged in research on one or more of the topics listed above.

This training program should greatly stimulate teaching and research in numerical analysis in universities. It complements the work being done at some universities and does not conflict with it since the universities are in the main concerned with the training of students and this program has as its trainees established mathematicians.

3. The Computation Laboratory (Section 11.02) like other sections of the Division, is having difficulty in maintaining its staff. If this situation is viewed as an educational service to the community in that it furnishes to industry and to other agencies of the government young people with good training in computing procedures, the Bureau may well take pride in the role it is playing.

However, the problem is a difficult one, particularly as the Section experiences a need for the expansion of its staff to provide for the increased work it must undertake with the delivery of the IBM 704 and the obligation to provide some programming help to DOFL.

The Handbook of Mathematical Tables is progressing well. The Section is to be particularly commended for drawing into the work experts in the special phases of mathematics that are represented either by securing the temporary services of these experts at the Bureau, or by contracting out the preparation of specific parts of the book.

4. Statistical Engineering Laboratory (Section 11.03) has as a major activity the consulting with scientists in other divisions of the Bureau on statistical aspects of their problems, including design of experiment and analysis of results. It is performing this function well, and its services are in great demand. The SEL continues to be a center of basic research in experimental design; ten research papers in experimental design are in process of publication in various technical journals. The reliability study recently undertaken by SEL has made a good beginning and promises to produce valuable results.

5. Mathematical Physics Section (11.04) carries on fundamental research on

problems of its own, and has the function of providing high level specialist consulting services to other parts of the Bureau in the field of mathematical physics. The committee finds that a good program of work is going forward within the limitations of the small staff available.

The idea of having a central consulting group on difficult problems of mathematical physics to serve the rest of the Bureau is sound and commendable. But it can only function effectively if its staff includes creative scientists of the highest caliber. By its nature the work cannot be carried on effectively by persons in the lower ranks. It follows necessarily that this section must develop in a way that its staff will have considerably higher than average grade ratings as compared with the rest of the Bureau of other comparable scientific laboratories.

The development thus far has been mostly in the direction of fundamental work on theoretical mechanics. In the years to come corresponding staff strengthening should occur to permit the rendering of effective consulting service in thermodynamics, statistical mechanics, chemical physics, electrodynamics, and so on.

During the past year important contributions were made by the section in exact non-linear water wave theory, theoretical elasticity, the theory of elastic surface waves, the theory of diffraction and reflection of electromagnetic waves, and the study of the Fourier transformations for distribution functions occurring in applied statistics.

The strength of the section in carrying on this program of highly original work has been greatly enhanced because the Bureau has been following a wise and highly praiseworthy policy of having on its staff a succession of some of the most renowned mathematical physicists in the world for varying periods as visitors. These have included Stoneley of Cambridge, England, Synge of Dublin, Eire, Schultz-Grunow of Aachen, West Germany, Burgers of Delft, Holland, and Walz of Göttingen, West Germany.

The field of mathematical physics is one that gains in creative productivity in unusual degree from stimulating contacts with great men in the field from other lands. The Bureau is indeed to be congratulated, in the opinion of the committee, on having developed such a first-rate stimulating atmosphere that men of such high distinction find it worth while to spend some time there as visiting research workers, and to give the Bureau the benefit of sharing their latest research ideas. This phase of the program represents a very genuine step forward which deserves every encouragement.

Since further recruiting will be required to maintain the research of this section at its present level of excellence, and to extend their consulting service to other divisions of the Bureau, it may be useful to review the philosophy behind the establishment of Section 11.04, to see how important its maintenance might be. Many Divisions of the Bureau deal with various aspects of classical (nonatomic) physics and, in order to improve their measurement techniques and broaden their service coverage, they require detailed theoretical computations in the fields of classical physics. Many of these computations are difficult enough to be classed as important research projects in theoretical physics, requiring theoretical physicists of exceptional ability to solve them successfully.

One way to get such problems solved is for each Division of the Bureau to hire

theoretical specialists in their own field, but this has several disadvantages. The number of challenging problems arising in any one Division may not be sufficient to attract the services of a first-class man and the general attitude of the Division may be such as to discourage theoretical research in new directions. Moreover the specialist, being isolated in one Division, may find it difficult to obtain the advice and encouragement from other theoretical workers, in other specialties, which often can make the difference between failure and success in some difficult problem.

Section 11.04 represents an alternate way of providing such specialized consulting services. The theoretical physicists are kept as a group, supporting each other in their work and able to divert manpower, as needed, to various important problems as they arise in the various Divisions. This arrangement utilizes scarce manpower more efficiently and makes it easier to attract good men, but it places a considerable responsibility on the head of the Section to keep in touch with the various Divisions of the Bureau so that the Section can be of real help to them (even, at times, to the extent of persuading the Divisions to get interested in an area which they had neglected before).

If the Bureau is convinced that the maintenance of Section 11.04 is the correct way to provide consulting services for the Divisions on difficult theoretical problems, it must be prepared to make special efforts to recruit the experienced personnel needed by the Section. A Section with these responsibilities, staffed by mediocre people, would defeat its purpose. If such recruiting involves the hiring of specialists from Europe, special arrangements should be made to make this possible.

Respectfully submitted,

A. H. Taub for the
Technical Advisory Committee
for Mathematics
David Blackwell
E. U. Condon
Mark Kac
Philip M. Morse
Mina Rees
A. H. Taub, Chairman

Index

In this volume, *MTAC* has continued its policy of trying to review all published tables, other than the most elementary ones, which depend on or contribute to computation. In addition, there have been reviews of works, essentially book length in size, contributing in some way to numerical analysis or its applications. A classified index attempts to furnish the reader with as much information as is conveniently summarized about these works. The index includes references to tables from number theory, algebra, analysis, statistics and physics, and to works or tables relating to numerical analysis, to the social and biological sciences, and to computing equipment and its use.

In the present volume, the author and the nature of the function tabulated

are put close together. This is a result of a suggestion by Dr. J. C. P. Miller and seconded by others. In the section devoted to analysis, the classification of the Fletcher, Miller, and Rosenhead *Index* has been retained without radical change. In the section devoted to statistics, an outline prepared by Professor H. O. Hartley has been used with considerable augmentation. Also, it should be noted that, while analysis tables are assigned fairly consistently to the class to which the tabulated function belongs, the statistics index frequently assigns a table to the section devoted to the use of the function. Some ambiguity in classification is unavoidable, and multiple entries are common.

In the tables from number theory, the classification of the Lehmer *Index* is followed without essential change (except for the addition of a section devoted to tables pertaining to analytic number theory not suitably assigned elsewhere).

Detail in the classification of the works on numerical analysis seemed justified by the amount of correspondence and conversation which exists concerning adequate presentations of various types of material; this has led to voluminous multiple listings.

The indexing was done largely by the Chairman of the Editorial Committee, but he gratefully acknowledges the help of W. J. Dixon, Rudolph Hüsser, and J. D. Swift—the first two in connection with the classification of tables from statistics, and the last in connection with number theory.

C. B. T.

CORRIGENDA

D. R. Morrison, "A method for computing certain inverse functions," *MTAC*, v. 10, 1956.

	for	*read*
p. 205, line −4,	$1/2(2^{\epsilon} - 1)$	$\log_2 (1 + 2\epsilon)$
p. 206, line 14,	f^{-1} f^{-1}	f f
p. 206, line −11,	f^{-1} f^{-1}	f f
p. 207, line −4	$\frac{\pi}{2}$ arc cos $(1/y)$	$\frac{\pi}{2} -$ arc cos $(1/y)$
p. 205, lines −3, −2, *omit*		It is less than $\frac{1}{2}\ \frac{\epsilon \log_e{}^2}{1 - \epsilon \log_e{}^2}$.

H. J. Hauer
E. A. Fay

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Review 24, *MTAC*, v. 11, 1957, p. 31, line 3 from bottom,
Exponent *should read r not t*.

VOLUME XI

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